Persian Variations

Jean-Marc Castera
6, rue Alphand, 75013 Paris, France
jm@castera.net

Abstract

The concept of variation is essential in geometric design. It is surprising that patterns very different may be variations of the same model.
We define two families of pentagonal patterns with three kind of variations, and give some suggestions how to analyse these patterns and create in this style.
We then search for self-similarity systems in a strict sense. Although from a systematic search, the two solutions proposed here can also generate some traditional 2-level patterns.
In searching for subdivisions of the tiles into rhombuses, we found two solutions. Both can be compatible with the Binary Tiling (not with the Penrose Tiling).
Then, using the concept of X-Tiles defined in a previous paper [3], we find new relationship between the two families of pentagonal patterns.
In the last chapter we show and comment some examples taken from traditional architecture in Iran, and infer a self-similar system for pattern with interlaces from a 2-level tiling in Isfahan.
This paper reflect the point of view of a pattern designer.

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1. Pentagonal Patterns

Terminology.
If everyone agrees more or less on what are the Pentagonal Patterns, it is not the case on what are their subfamilies, and how to name them. In this work I have to define subfamilies according to their components, name the tiles, and name some self-similar systems. For subfamilies, even if they are somehow similar to other defined by AJ Lee, J. Bonner, P. Cromwell [11, 1, 2, 7] and others, they do not have the same origin. That's why I have sometimes to use some specific names. Eventually, it is in the Iranian tradition that I found a classification of patterns close to the one proposed here. I will use it also, specially in the last part of this paper.

Two families of Pentagonal Patterns.
Let {S} be the "Starry Family" of pentagonal patterns, made of the set [S] of shapes derived from the decagonal star {10/3}, and {F} the "Flower Family", whose main element is the standard 10-rosette. The first family is typically Persian, the second can be seen in any place of the historic world of Islam. We start talking about the Starry Family, and will meet the Flower Family in chapter 6.

A set of tiles generator of the Starry **family {S}.**
Let it be [S]. All the angles in this set are multiples of 36°. The patterns made from it can be considered basic. They are usually subject to variations, that can generate new tiles which we consider as secondary. We will see in chapter 4 that this set [S] have been defined from the search of self-similarity systems. The sub-set [S1] contains the 5 most common tiles, and have some remarkable properties.

First remark, there are **two kind of tiles**: the Positive Tiles, P1, P2,... P5, and the Negative Tiles, N1, N2... N6. When the pattern is colored like a chessboard, every Positive Tile is Black, while the others are White (or the contrary).

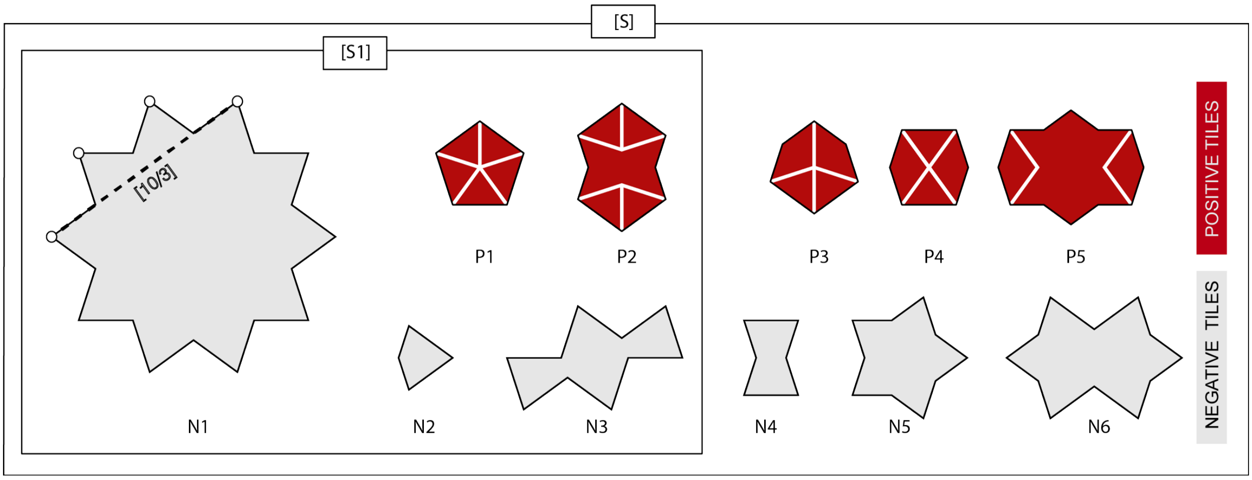


Figure 1. The set of tiles [S] used in the "Starry" family of patterns, {S}. In red the Positive Tiles (with addition of the "Polygonal Lines"). In light the Negative Tiles. N1 is "the mother" of this set. On the left, the sub-set [S1].

The additional lines, drawn on the Positive Tiles, are used only to make a link with the PIC method. Let them be the "Polygonal Lines".
The tile N1, which is the {10/3} star, can be considered the mother of that set, although it is typically subject to variations. Those variations enrich the pattern, but can hide the main structure as well.

Note: That classification is not far from the one used by traditional artists in Iran [14,19], who consider 3 families of tiles: "*Kond*" (which correspond to [S1]), "*Shol*", and "*Tond*" (tiles of the second family, {F}, of pentagonal patterns. See components in Figure 30). The set [S] correspond more or less to what they call "*Kond* and *Shol*".

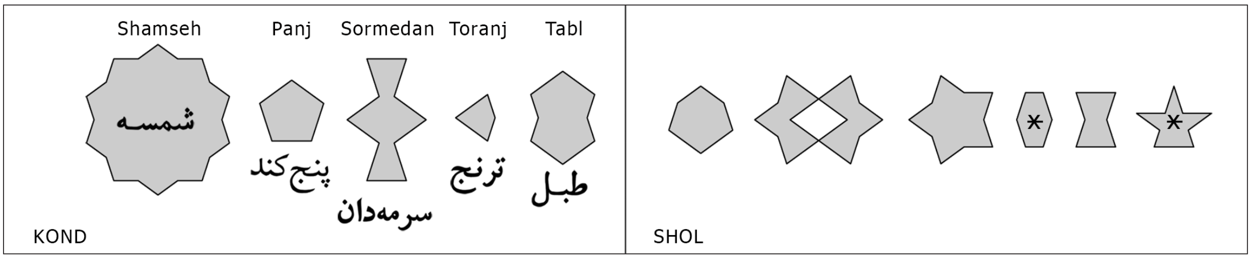


Figure 2. "*Kond*" and "*Shol*" sets of tiles, from Iranian traditional classification. The *Kond* family correspond to [S1] even though the "mother" N1 is missing (because always decorated with a {10/2} (*Shamseh*) inside). N6 is decorated with a rhombus, which correspond to the variation $N6_1$ (Figure 3), and we have two extra *Shol* tiles (✻) compare to [S].

Rules for assembling the tiles. It's obvious, but better to say: Tiles are put together in respect with the continuity of the edges. Thus, tiles of the same kind (Positive or Negative) are never connected by edges. We could use only the Positive Tiles connected by vertexes (that's the simplest way to work with computer vector softwares). In that case, the shape of the holes are the shapes of the Negative Tiles.

First kind of variations: When one tile is replaced by a set of tiles. Below, the variations on N1 and N6:

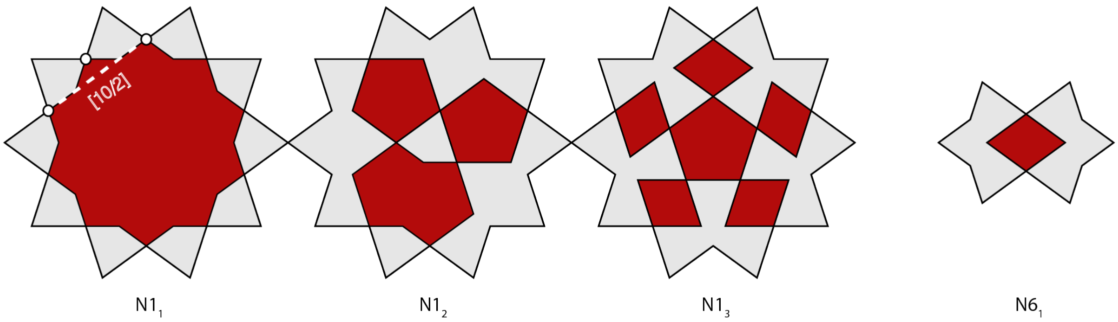


Figure 3. Variations on the "mother", N1. $N1_2$ contain every tiles of [S1], while $N1_1$, $N1_3$ and $N6_1$ generate new tiles

Each variation, except $N1_2$, generates secondary tiles. $N1_2$ include all components of [S1]. Sometimes it's not that obvious to recognize the hidden N1 in a complex pattern (This can be a good exercise).

Second kind of variations: when a connected set of tiles is replaced by another one.
The examples below (Figure 4) introduces an element of the flower family {F}, the famous "10-rosette", into the Starry Family {S}.

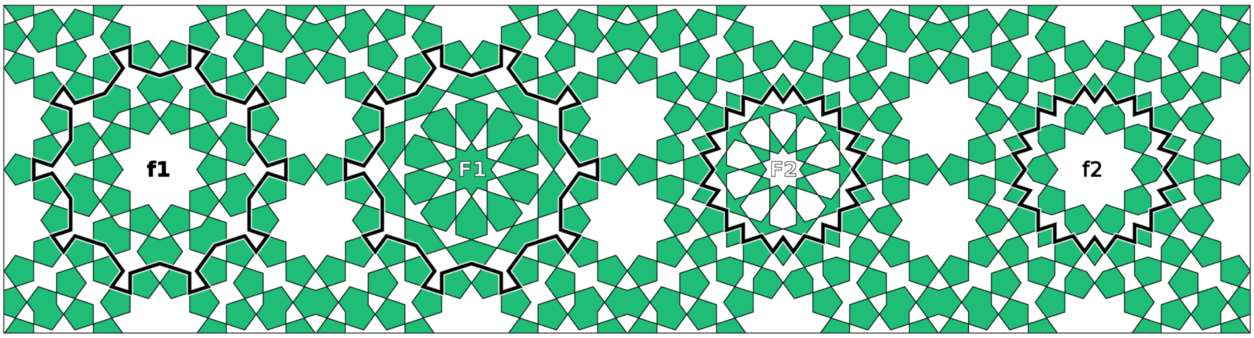


Figure 4. Two different ways to incorporate the 10-rosette into the Starry family. F1 takes place of f1, F2 takes place of f2.

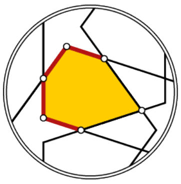


Figure 5. The 4 equal edges of the rosette's petals.

Here two versions of this rosette, F1 and F2, are incorporated differently into a pattern of type {S}. Although the scales are different, the proportions are the same (with 4 edges equals, the hexagonal petals are as regular as possible). F1 can be seen on the famous Karatay panel in Konya (Figure 13). The other one...in many places. Note the color inversion between F1 and F2.

Third kind of variations: when a transformation is applied to the whole pattern. That definition include also the case of multilevel and self-similar patterns discussed in chapter 4.

The two examples (Figure 6), from the Iranian tradition [14], show how to get patterns with rosettes from the two simplest *Kond* patterns K1 and K2. The first one (named "the mother of tilings" in Iran) gives way to the simplest pattern with 10-pointed rosettes, which can be seen everywhere in the world of Islam. The PIC method would give the same result in a different way. In the second case, the rosette has two types of petals.

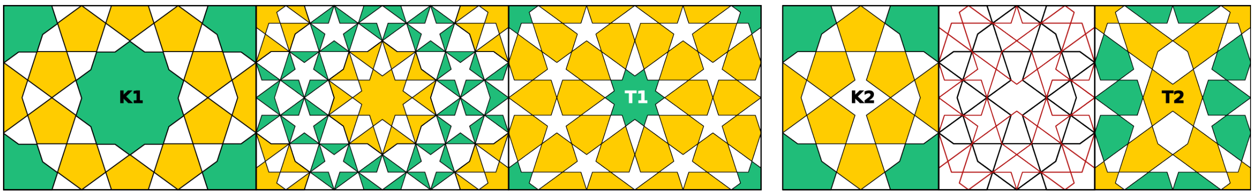


Figure 6. Traditional transformations from a "*Kond*" pattern (K1, K2) to a "*Tond*" one (T1, T2).
The second case generates irregular rosettes at the corners.

Notice to the reader: in order to always consider the patterns in their simplest form we do not use variation on N1 (the mother) in the following drawings, even though in the real world of traditional patterns of the Starry family {S}, the mother is always "pregnant" (always has something inside).
The reader is encouraged to design patterns in that way, and to introduce variations afterwards.
But what are these Polygonal Lines, drawn on each Positive Tiles (Figure1)? Now, I have to talk about the PIC method.

2. The point of view from the PIC method

The Polygon In Contact (PIC) method proposed by Hankin in 1925 [8], improved by C. Kaplan and J.Bonner, can be used for constructing tilings, starting from an underlying polygonal pattern.
In the simplest -and more elegant- case, the polygons are equilateral. They are decorated by lines issued symmetrically from "Crossing Points" at the middle of each edge with a constant angle, running straight until they meet together. When the pattern is done, the polygonal lines are erased and it remains the actual tiling.
The qualities of the model have been widely praised in countless publications, so I would like to mention here some of its minor flaws. Of course, the reader unfamiliar with the PIC theory can skip this chapter.

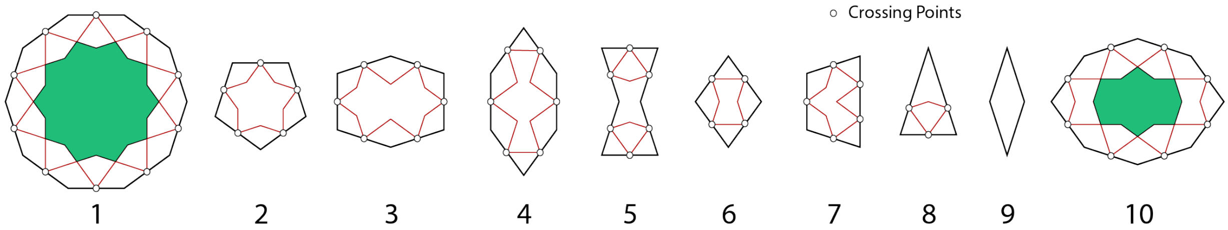


Figure 7 Some polygons in use by Bonner, and their decoration in the case of "Median Design" [1,2].

1. In terms of pattern analysis, finding an underlying polygonal pattern do not answers to the question of how the later have been conceived and constructed.
In term of creation, there is no evidence that it is easier to work with such polygons than directly with the actual tiles.
2. The decagon (Figure 7-1) is always shown with a {10/2} star, or "Shamseh", inside (variation $N1_1$ in Figure 3). To be more consistent, it would be appropriate to add a rhomb in polygon (3) (variation $N6_1$).
By the way, and more important: the "mother" N1 is hidden.
3. The first set of equilateral polygons (1 to 6) is elegant. But, in order to work with more patterns, some non equilateral polygons are needed. Some crossing points are not at the middle of the edge (8), polygon (7) is "hybrid" (one edge having two crossing points on it), polygon (9) is undecorated and sometimes (not shown here) several decoration are used for a same polygon. May be effective, but not that much elegant.
4. The economy in terms of number of elements is poor. And if we consider that any pattern made with the set of tiles [S] could be made with only the Positive tiles connected by vertex, there is no economy at all.
Example: every pattern made from the set [S1] can be constructed with only the two Positive tiles of the set, while two or three polygons are needed with the PIC system.
5. Some patterns with elements from the two systems {S} and {F} cannot be produced -directly- from the PIC modules.
7. Some scholars find strong evidence in the use of PIC method by traditional Persian artists [1, 6,7, 12, 13]. That conventional view relies on a questionable interpretation of the Topkapi Scroll [15]. Moreover, I did not find any mention of the PIC method it in the Iranian books of tiling I know [14, 19, other cited in 18].
6. **The relationship between the polygonal and the actual patterns is not necessarily a causal relationship, this is nothing more than a kind of duality.** That's the reason why I have drawn the "Polygonal Lines" on the Positive tiles: From any pattern made from the set of tiles [S], those lines define automatically the "hidden" polygonal pattern, after erasing the orphan lines (Polygonal Lines having one vertex unconnected to another Polygonal Line).

Example.
Let's start with the pattern previously designed for the variations with 10-rosettes.

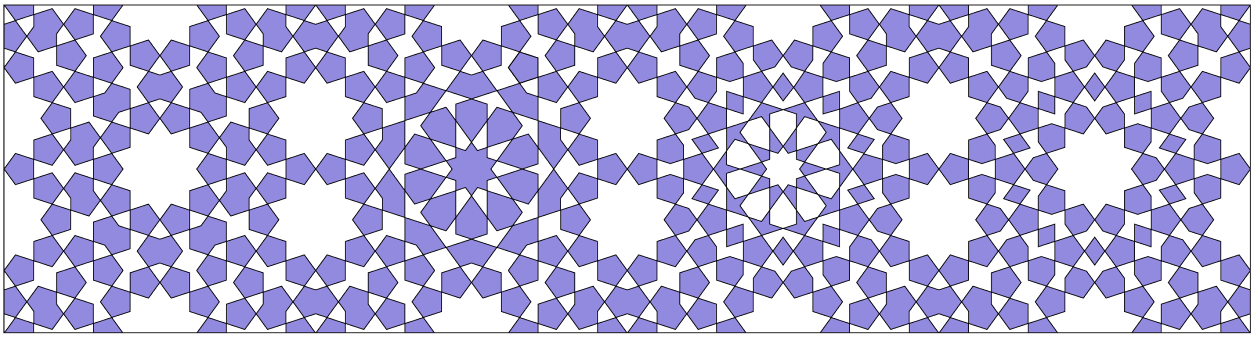


Figure 8 The pattern (Made from the set [S1] with variations $N6_1$ and rosettes).

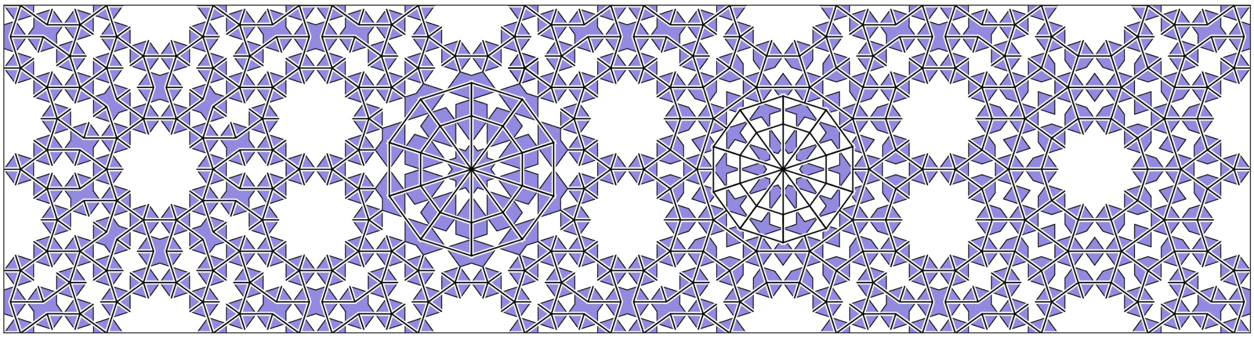


Figure 9 With the Polygonal Lines drawn on each Positive Tile. Remove the orphan lines...

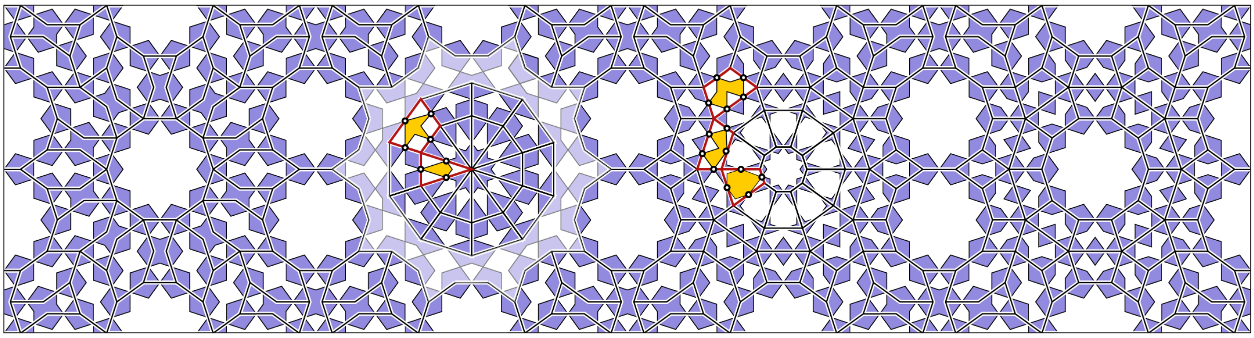


Figure 10... And here is the polygonal pattern of the PIC method. The additional polygons are highlighted in bold.

Note that:
1. The PIC method doesn't works properly for the connexion zone around the left rosette.
2. Five non equilateral polygons have to be define. Two of them are "hybrids" and the pairs of lines don't make always the same angle.

Anyway, some contemporary artists in the field are working with this technique, producing very good works (J. Bonner, M. Pelletier [2, 17]). Several different methods can be used, each having its qualities and flaws. In another publication [3], I've imagined a morphogenesis of pentagonal patterns. I do not claim that it has been used at the historical period, but it works as well.
However, there is at least one case where this technique could be essential: The case of the 2-Points family, when there is not one but two connecting points on each edge [1, 2,9, 10].
An interesting thing is the possibility to produce different patterns by changing the angle of the pairs of lines (What can be done interactively with C. Kaplan's Taprats software [9. see also 1,2]).

Note that we could similarly define some polygonal lines on the negative tiles, like on the following image. This generates a new pattern, similar to those used for mashrabiyya.

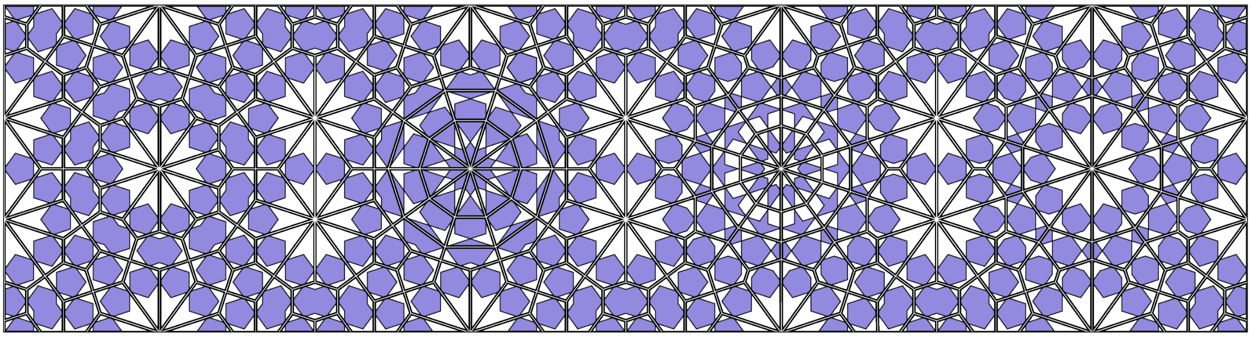


Figure 11. Polygonal lines drawn on the negative tiles

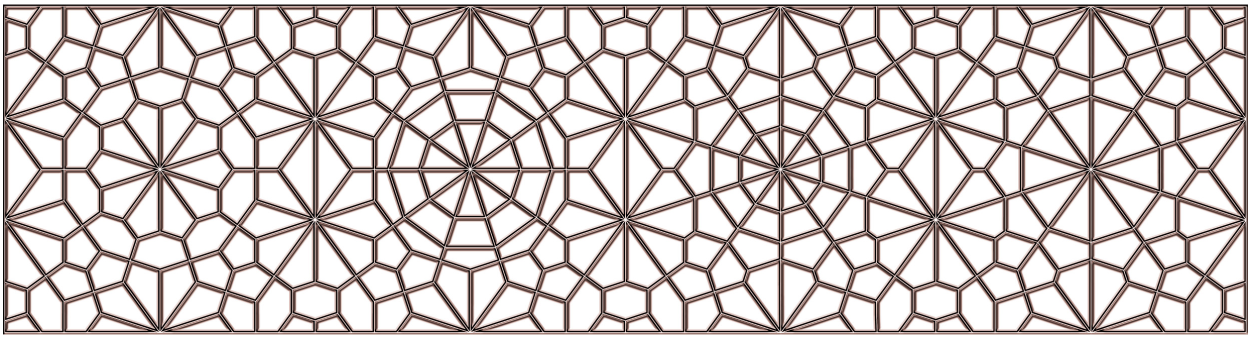


Figure 12. Same pattern with only the negative polygonal lines

The variation which consists to include the 10-rosette is an example of connexion between the two families {S} and {F} of Persian pentagonal patterns. We are going to meet other surprising connections in chapter 6.

3. Suggestions for analyzing a pattern of the starry family {S}

The key words are: simplification and main lines. After analyzing the symmetries, try to find the other main construction lines (like what you do when sketching from life). Then try to identify what is not essential, where are the variations, in order to get the simplest basic pattern.
As an example, let's consider two famous patterns: One from the Maragha blue tomb, Iran, the other one from Karatay medersa, Konya, Turkey.

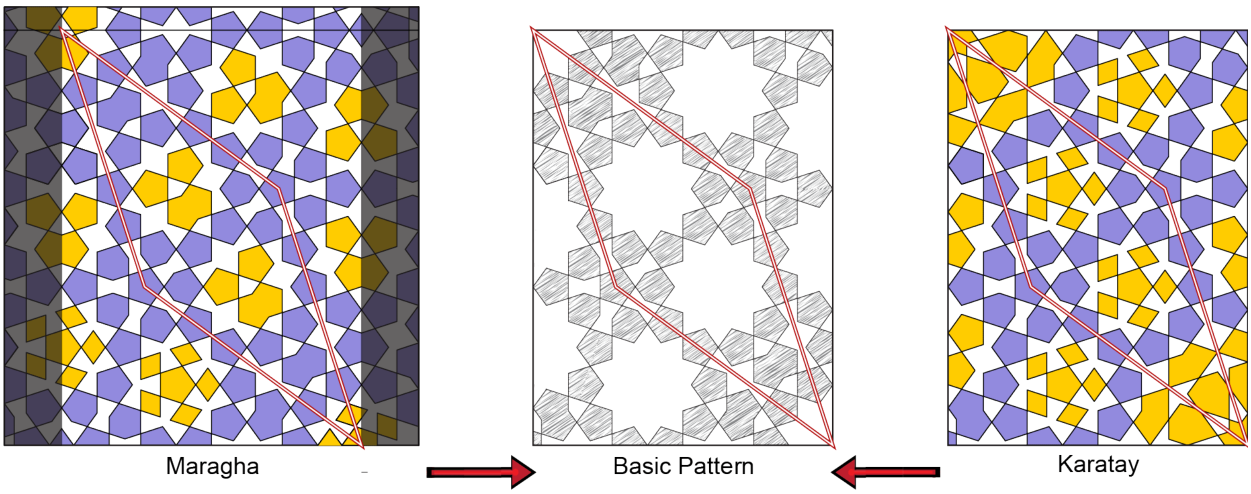


Figure 13. Once all variations (colored in yellow) have been identifyed then removed, these two famous patterns can be seen as different interpretations from the same basic design (middle). Left, Maragha blue tomb, Iran. Right, Karatay medersa, Konya, Turkey.

I've drawn the entire Maragha panel (left), and reduced the other to 1/4 (right). The variations ($N1_1$, $N1_2$, $N1_3$ and rosettes) are colored in yellow. Both units have the same structure: a great lozenge on the diagonal. Then, after reducing the patterns to their simplest expression (removing all variations), it remains exactly the same basic pattern (which comes from the 4th generation in my morphogenesis process [3]). That surprising fact would be difficult to see without any simplification.

Experimentation.

The kind of representation needed differs according to what you are doing with patterns: Analyse, conception or making.

Conception: It makes sense to work directly with the tiles. Although there is no evidence, I can imagine that historical designers could have used paper tiles models to aid in the creation. In Morocco, some masters use directly the Zellij tiles [16, Vol. 1, p. 396], or different techniques [5] (but not the PIC method). Moreover the artists, as the mathematicians, can have a right reasoning from a bad figure.

In terms of making, it depend of the technique. If woodwork or plaster needs a drawing or a stencil in real size, this is not the case with the ceramic technique, where the tiles are cut individually from a template, and put together afterward on a plane surface where only the symmetry lines are drawn. The template for each kind of tile has to be as perfect as possible, but this needs nothing more than elementary geometry. When a drawing is needed, it don't need to be geometrically perfect. For example, no sophisticated geometry is needed to divide with a compass a circle into 5 or 7 sufficiently equal parts: simply try and correct.

The above photos show the first game I did to study a sub-set of {S} by experience. There are only 3 Positive tiles (A new version includes all the tiles of the set [S]), with the Polygonal Lines engraved. The corresponding polygonal pattern appears naturally when we play with the puzzle, but here this is not the cause but the consequence. With this set of tiles, we can make non-periodic patterns as well as periodicals.

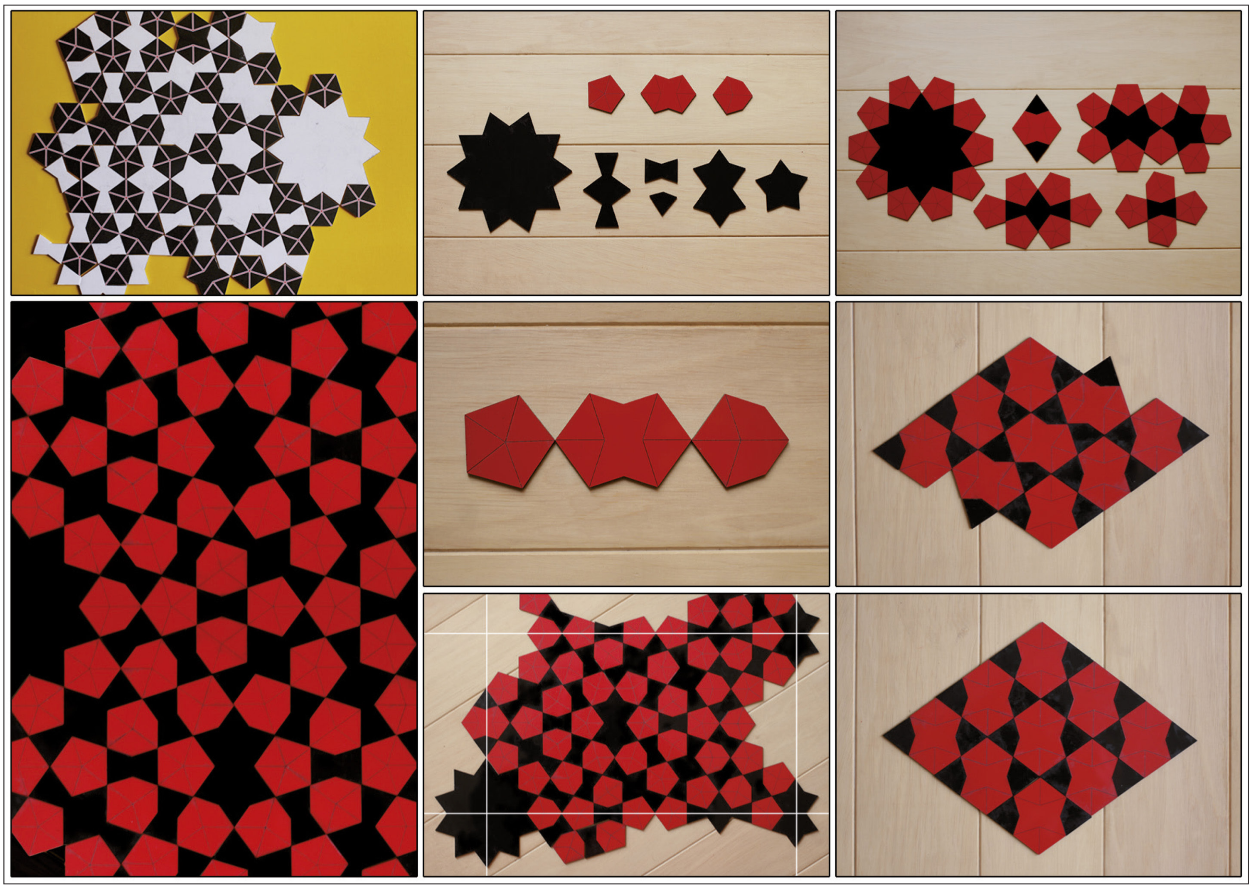


Figure 14. Top-right: Connected by vertices, respecting the continuity of the edges, the Positive Tiles (in red) generate the Negative Tiles (in black). So, we basically could use only the Positive Tiles... with the holes in between.

4. Two Self Similarity Systems on {S}

In this research to self-similarity systems I consider the actual tiles of the patterns, never the associated polygons.
I consider the simplest version of the patterns, without variations. Thus, “the mother” of the tiles, N1, will be seen in its pure beauty...

Although they come from a systematic study, and not from the historical patterns, it comes out that the first solution is related to the 2-level patterns of the Topkapi Scroll [15], and the second to many patterns that can be seen in Isfahan (see section 7.5).

Each solution is characterized by the scale ratio, k, between two successive levels (so $k > 1$).
I could not get a solution between the two which we are going to discuss here, and haven’t searched that much on solutions with an upper scale ratio… until I met a special pattern in Isfahan (chapter 7-6).

Definition.
We call self-similar pattern (in the strict sense) a pattern made from a limited set of tiles, with the following properties:
1. Each tile of the set can be patterned with tiles of the same set at a reduced scale.
2. Applying these transformations to each tile of the pattern produces a new pattern without any gap or overlapping[1], so the process could go infinitely, level by level.
3. Any level can be mapped onto the next level (not onto itself because we are considering real patterns, which are finite).

Here we are searching for solutions in which each different shape is associated to one single transformation rule. These rules are named “Inflation Rules”.
In terms of visualization, 2 levels are enough, under the condition that the transformations are defined for every tile at the first level (with no extra tile at the second level).

According to this definition, no drawing of the Topkapi scroll is self-similar. I have not yet recognized a self-similar traditional pentagonal pattern, but only 2-level patterns ([15] pl. 28, 29, 31, 32, 34, and chapter 7 of this paper). Some may be very close, like at Darb-e Imam, Isfahan, but there is always something missing.

Peter Cromwell have recognized a scale-invariant design in the hexagonal family of Indian style [6].

Peter Lu and Steinhardt [12] state that the famous Darb-e Imam shrine pattern (Figure 57) is self-similar, according to the PIC point of view. But the first level pattern would be made from 2 polygons, while the second level needs one more (the hexagon). One inflation rule is missing, and the first level -which is periodic- cannot be mapped onto the second.

Method.
We first look for a way to pattern the edges of each shape with reduced tiles, and then try to fill the interiors.

1 To be exact, we would have to add a condition: either to accept a perfect superposition of some second level tiles along the edges of the first level, or to use a solution like in Figure 16, with cut tiles.

First solution, minimal scaling (ratio maximum).
The scale ratio is equal to $3 + \sqrt{5}$.
There is a strong similarity to the Topkapi Scroll 2-level patterns.
It works for any pattern made from the set [S]. In fact, this set has been defined from this process.

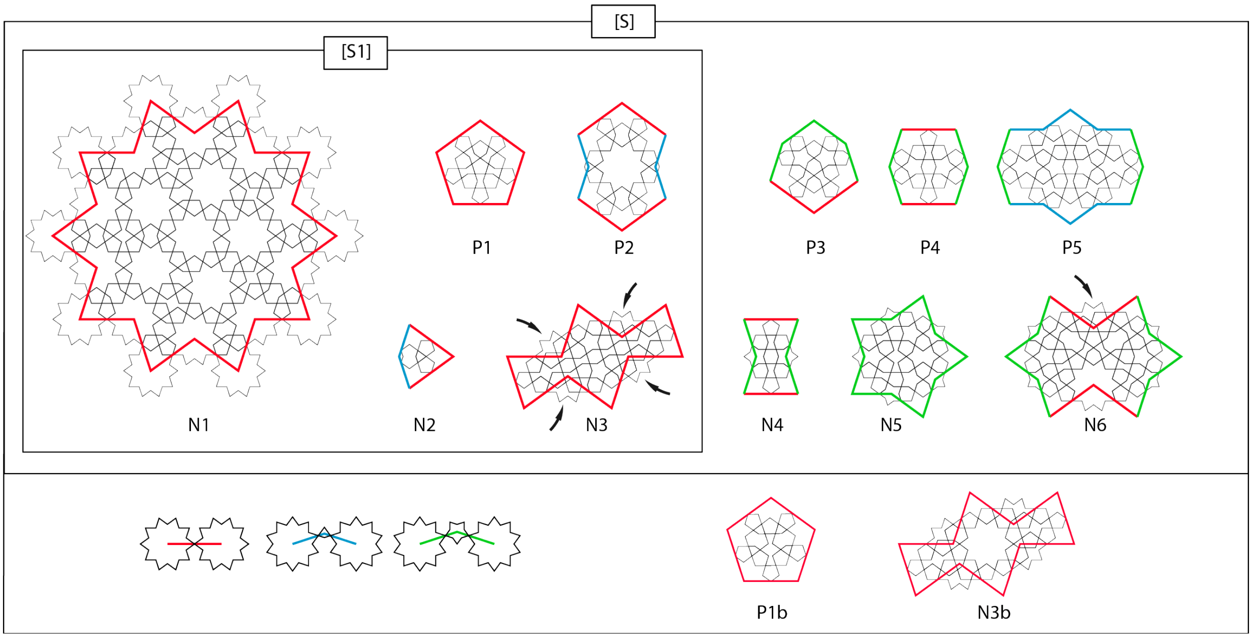


Figure 15. First system of self-similarity, "in the style of the Topkapi Scroll". It comes from the mapping of the 3 different kind of edges (bottom, left). Each tile of the set [S] is mapped by reduced tiles of the same set. P1b and N3b are alternative solutions for the tiles P1 and N3.

This mapping of the set of tiles [S] is strictly self-similar, even if we use the mapping P1b in place of P1 or N3b in place of N3. But not the both: in this case, the sub-set [S1] himself would be self-similar. You can imagine the consequences...

Take care to the mapping of N3, with its "little wings under the armpits" (arrows). When connected to a Positive Tile those wings break the usual star centered on the vertex. So we will not have to define different configurations for each positive tile. Same thing with N3b and N6.
Could we make a puzzle in real from that set? Certainly (Figure16), but the shapes are a little bit strange, as you can see.

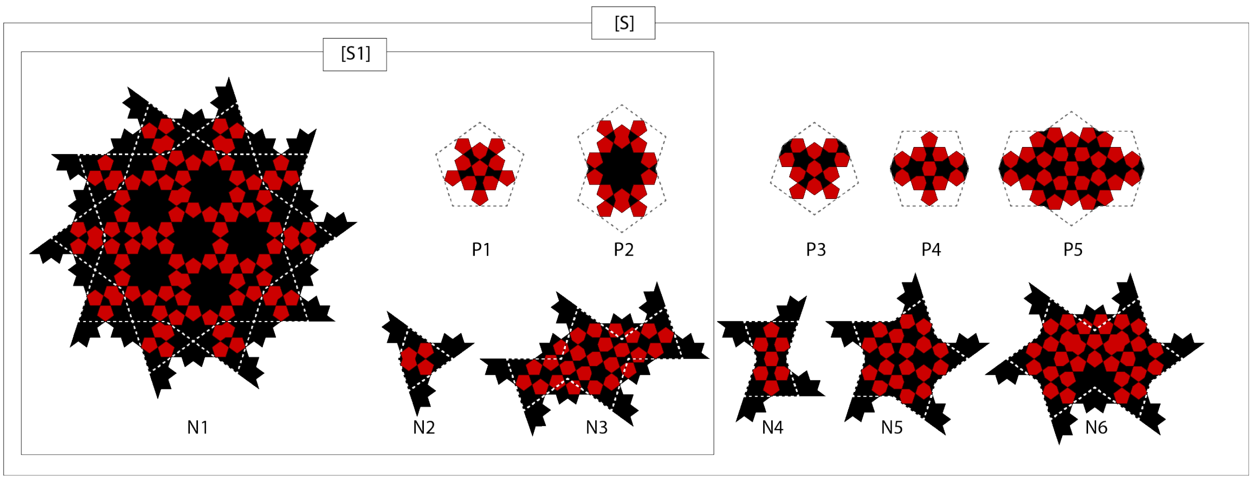


Figure 16. Proposal for the making of real multilevel tiles that can be put together without any problem of overlapping on the edges.

Second solution, "Isfahan Style".
This solution do not works for the complete set [S], but only for the sub-set [S1].
The scale ratio is equal to $4 + 2\sqrt{5}$.
There is a strong similarity to some 2-level patterns that can be seen in Isfahan, Iran.

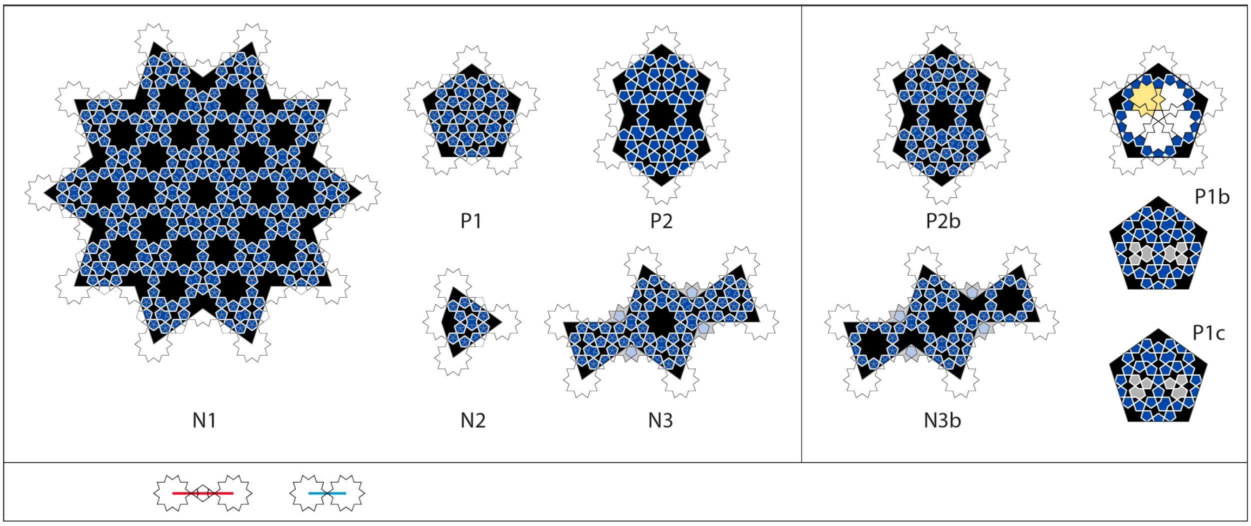


Figure 17. Second system of self-similarity, "in the style of Isfahan". It works only for the subset [S1]. It comes from the mapping of the 2 different kind of edges (bottom). Each tile of [S1] is mapped by reduced tiles of the same set.
Right: P1b, P1c, P2b and N3b are alternative solutions for the tiles P1, P2 and N3 (not exhaustive).

At the left, some other options (not exhaustive) for paving the tiles P1, P2 and N3. Again, the tile N3 needs "little wings under the armpits" because it cannot be a complete star here.

If we start the inflation process from any tile, the second level will always include the complete set [S1]. So, at the third level every transformation rule will have been defined and the pattern will be self-similar in the strict sense. At infinitum, it becomes a kind of 2D-quasicrystal.

The second-level elements are, for each tile:
With the first solution
P1 => P1, P3, N1, N4 ; **P2** => P1 , P2, N1, N2 ; **P3** => P1, P3, N1, N4, N5 ; **P4** => P1, P3, P4, N1, N4, N5 ; **P5** => P1, P3, P4, N1, N3, N4, N5, N6 ; **N1** => P1, P2, N1, N2, N3 ; **N2** => P1, P2, N1, N2 ; **N3** => P1, P3, P4, N1, N3, N4, N5 ; **N4** => P1, P3, P4, N1, N4, ; **N5** => P1, P3, N1, N3, N4, N5, N6 ; **N6** => P1, P2, P3, P5, N1, N2, N3, N4, N5.
Depending of the starting tile, we get the complete set [S] decorated at level 3 (from P5 or N6), 4, or 5 (from P2, N1 or N2).
With the second solution, we have the complete set [S1] decorated at the third level for each tile.

Next pages, three generations of the first system are applied from N3b (Figure 18) and N6 (Figure 19), so we get a 4-level pattern. Note that, to avoid falling into the "[S1]-attractor" we have not used P1b but P1. Two generations of the second system are applied from the tile P2 in (Figure 20). The next figure shows the same pattern drawn with only the Negative Polygonal Lines defined above (Chapter 2, Figure 12).

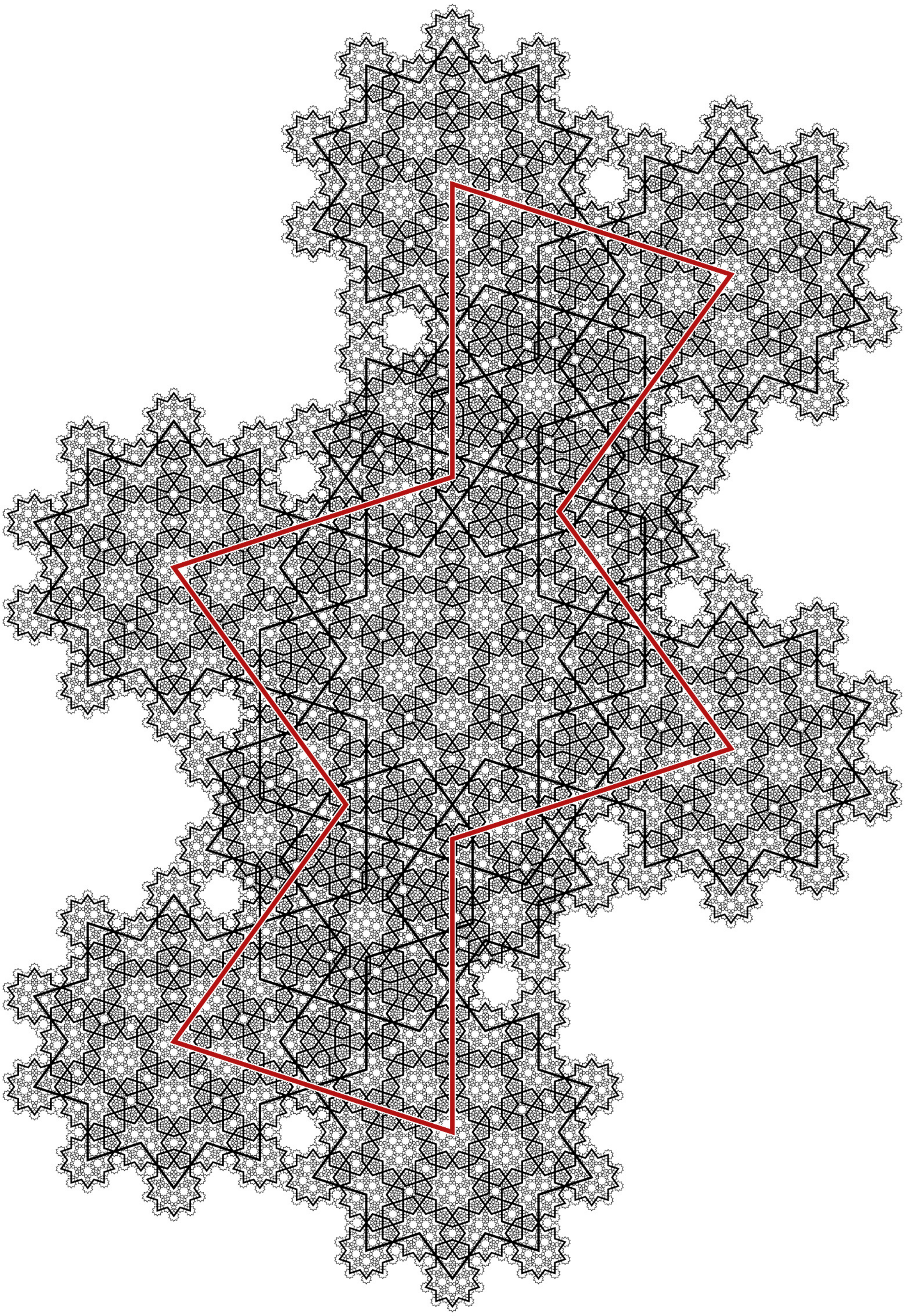


Figure 18. The first system of self-similarity is applied to the tile N3 with the mapping option N3b. Four levels are drawn on the figure. The two first levels belong to the family {S1} (tiles P1, P2, N1, N2, N3). P3 appears at the 3rd step, N5 and N6 at the 4th. Two steps are missing for having the complete set of mapped tiles. That slow process is due to the use of N3b in place of N3.

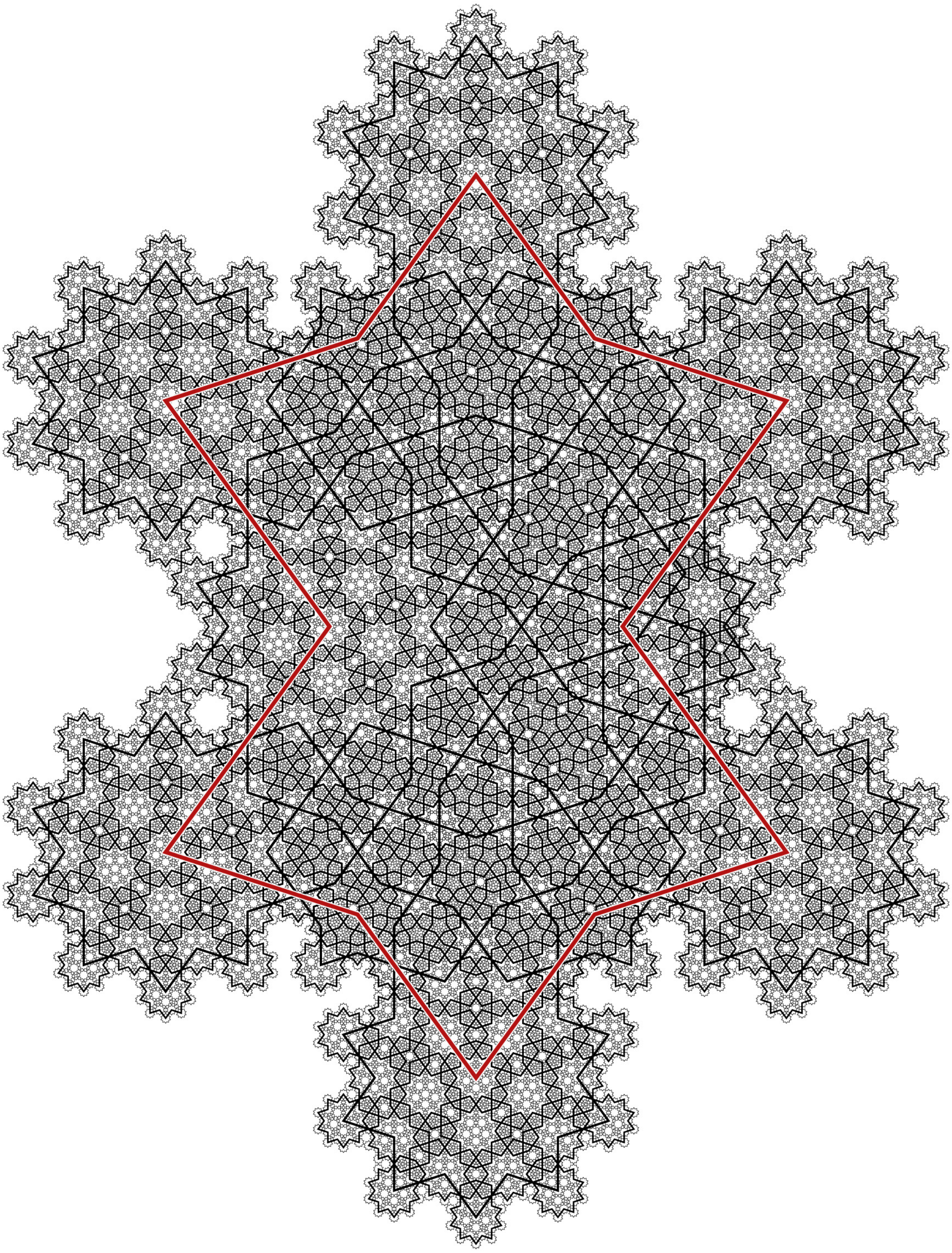


Figure 19. Starting from the tile N6, we get the complete set of mapped tiles at the 4th level.

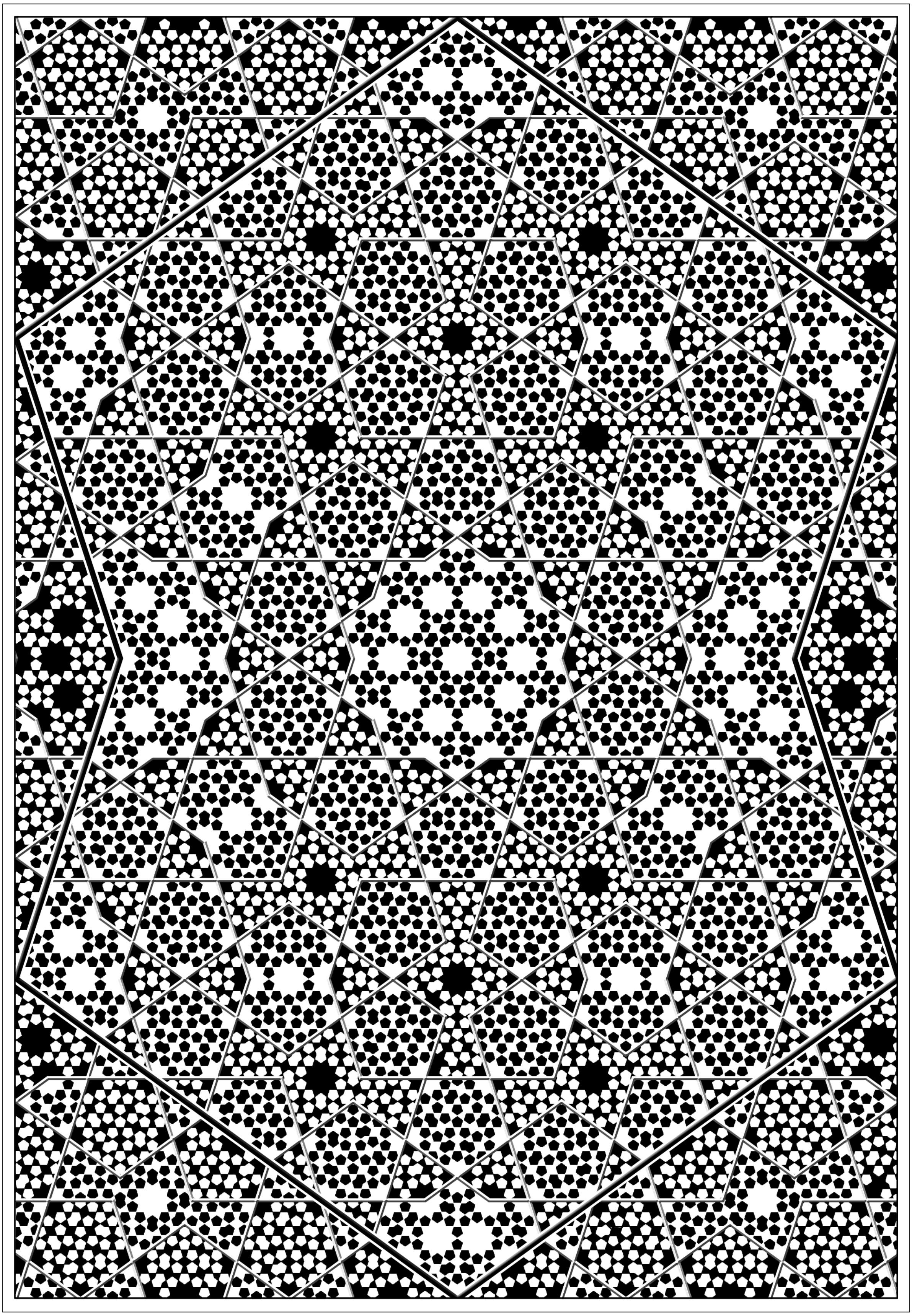


Figure 20. The second process of inflation (Isfahan Style) applied twice from the tile P2. The pattern has been incorporated into a rectangular frame, so it could be repeated after (some local adjustments along the vertical edges). We have used color inversion in order to distinguish the tiles at each level, although using only two colors.

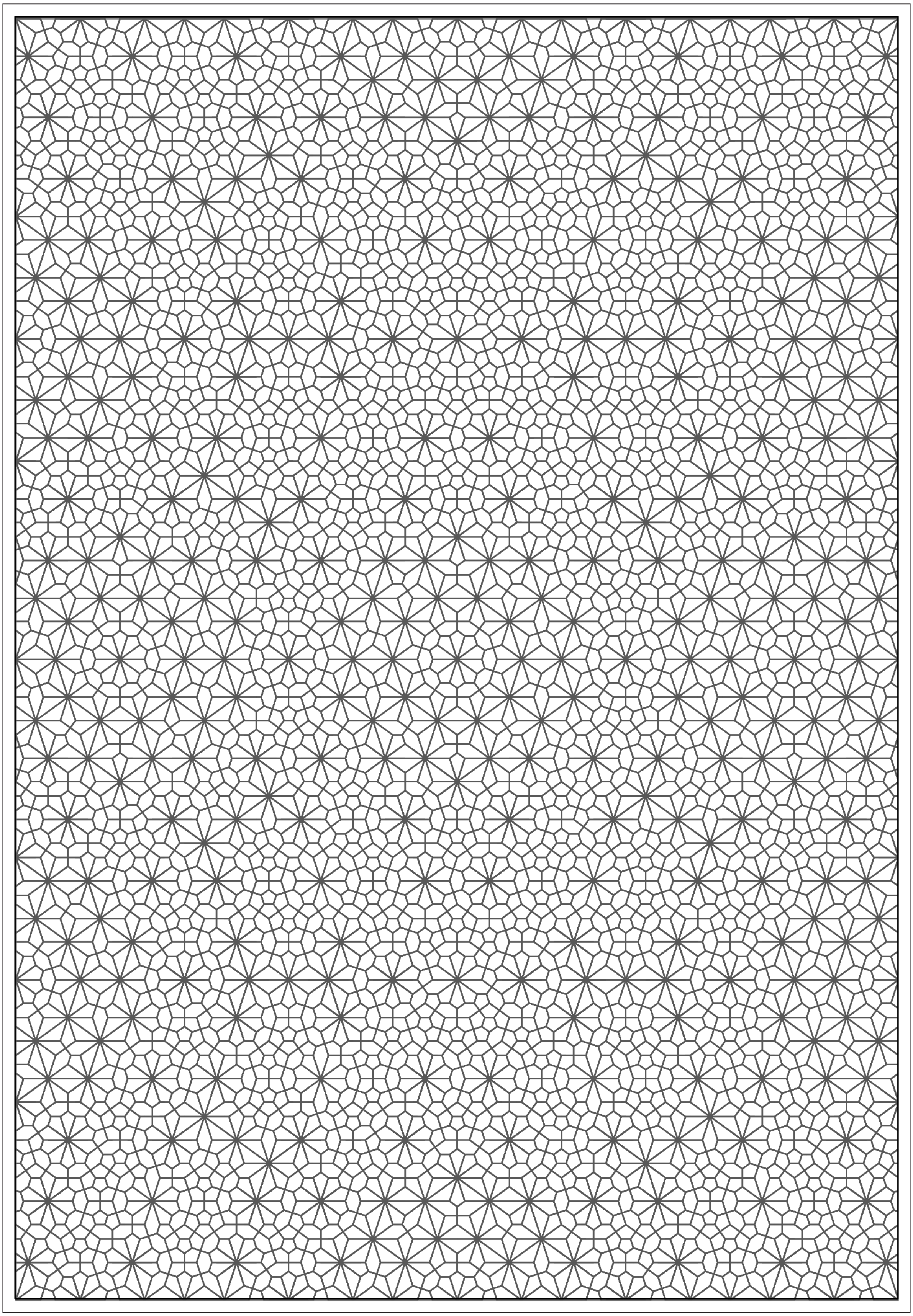


Figure 21. Here is the same pattern with only the polygonal lines drawn on the Negatives Tiles, as defined previously. Mashrabiyya style ?

5. Tiles decomposition into "Penta-Rhombs"

In a previous work on the octagonal system [4] it came out that each tile of a certain sub-set of the elementary elements of Zellij can be made from the two components of an Ammann tiling, which are a square and a 45° rhomb. Now the question is: could we define such a process for the pentagonal family? More precisely, for the set [S]?
I've found 2 solutions. The first works only for the sub-set [S1] while the second works for the whole set [S]... with some special conditions.
Obviously, the rhombs in the pentagonal system have to be the two famous used in a Penrose pattern. Let them be the "Penta-Rhombs".

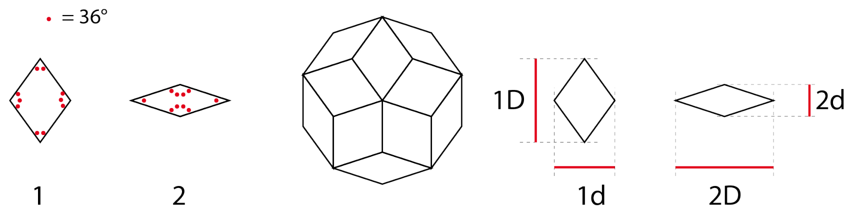


Figure 22. The two Penta-Rhombs mapping the decagon (center), with indication of the angles (left), and of the lengths (right). For each tile (1 and 2) we note d the short diagonal and D the large one.

A first solution.
The method starts by searching for different ways to map the edges of the tiles with the short (d) and long (D) diagonal of the rhombs. Then, trying to extend the mapping to the whole tile.

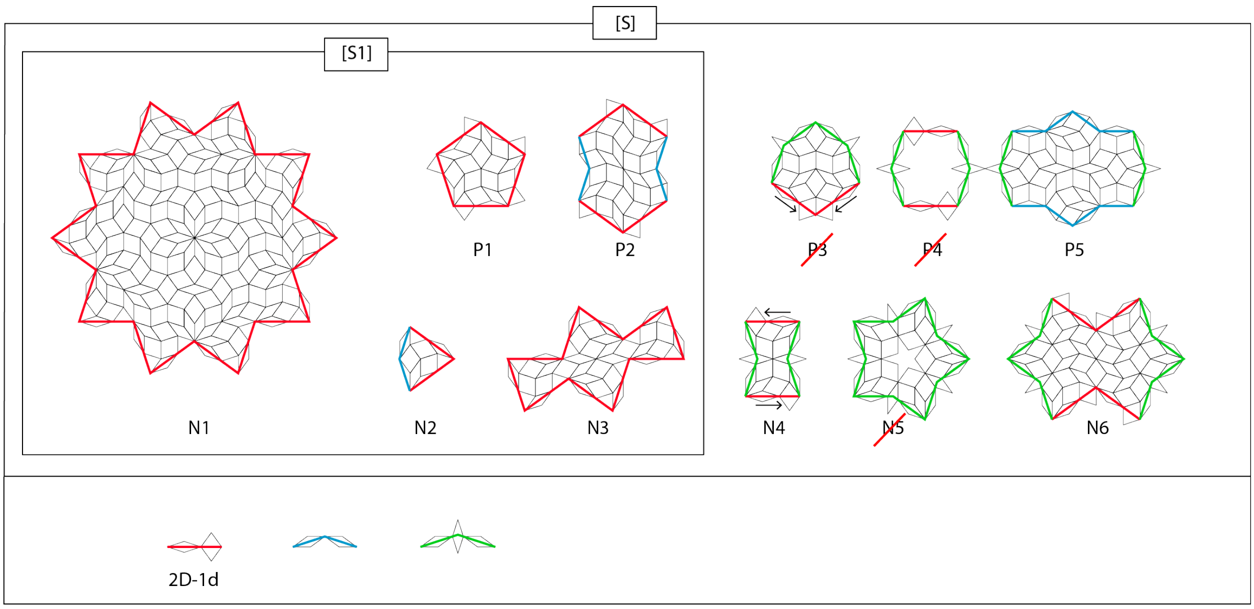


Figure 23. First solution for mapping the tiles with the Penta-Rhombs. It cannot works for P3, P4 and N5. Bottom, the mapping of the three different edges.

This solution can be define by the mapping of the edge of the pentagon: 2D-1d (long diagonal of 2 and short diagonal of 1). Unfortunately, it doesn't works for every tile of [S], but it works perfectly for [S1].

A second solution.

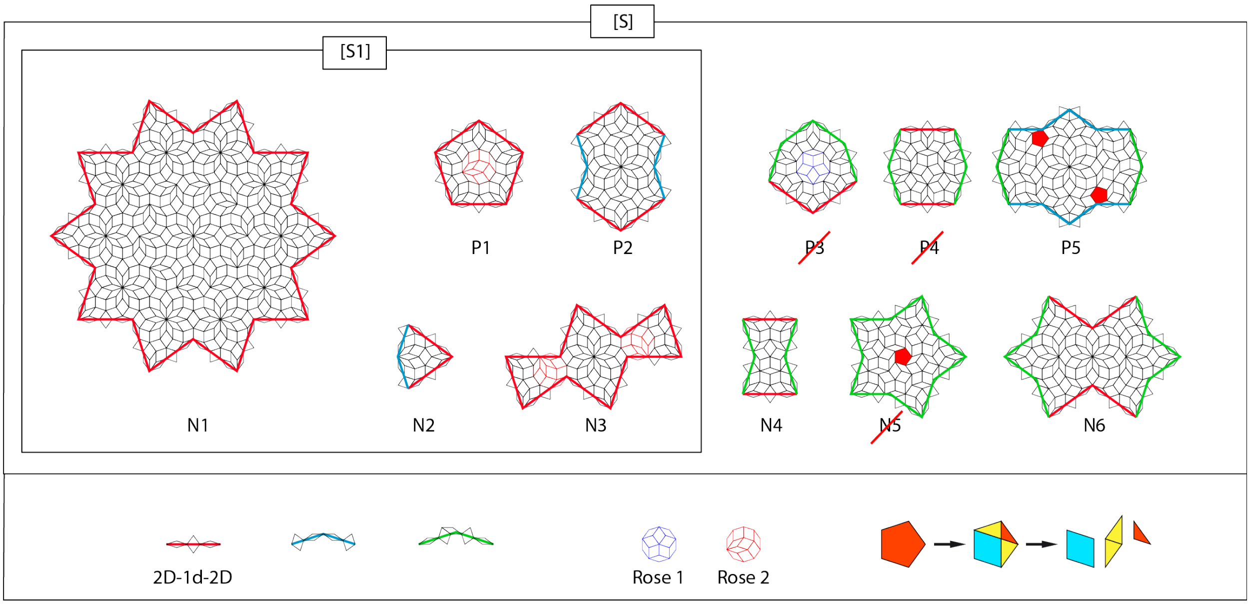


Figure 24. Second solution for mapping the tiles with the Penta-Rhombs. Bottom left, the mapping of the edges.

That one works better (Figure 24), but the mapping needs some extra pentagons for the tiles P5 and N5. Bottom right, we can see that there is no way to solve the problem, because the pentagon cannot be entirely broken into rhombs.
Anyway, the mapping works perfectly for the set [S1].
Note the two options for the same area, Rose 1 and Rose 2. On P1 and N3 we have chosen the second option. The reason will be given in the next chapter.
A slight variation of this solution is given on section 7.4 (Figure 53).

Next page, the two mappings are applied on the same pattern, the one we have previously used to illustrate the second system of self-similarity.

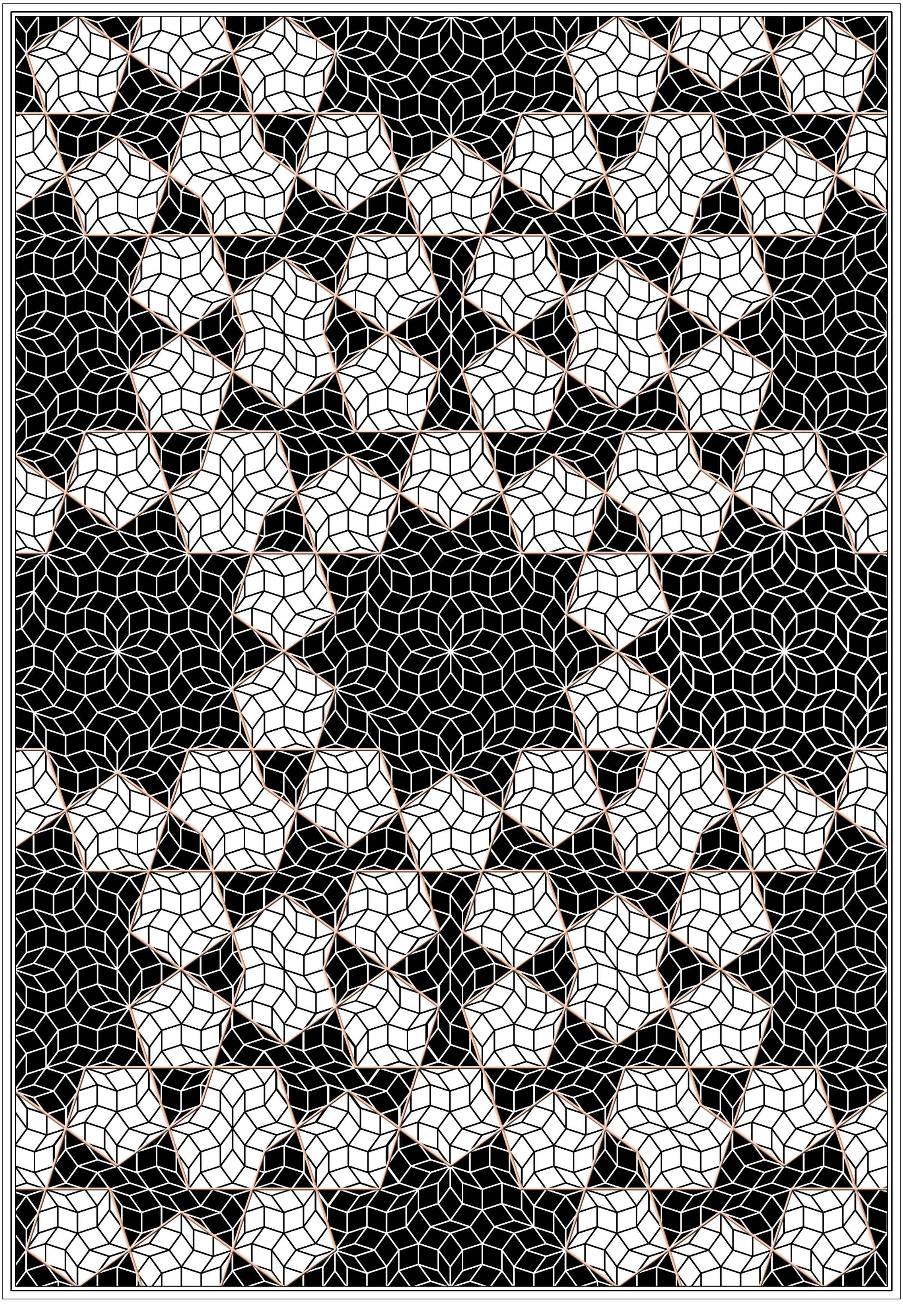


Figure 25. The first solution of mapping the tiles with Penta-Rhombs, applied on a pattern made of tiles from [S1].

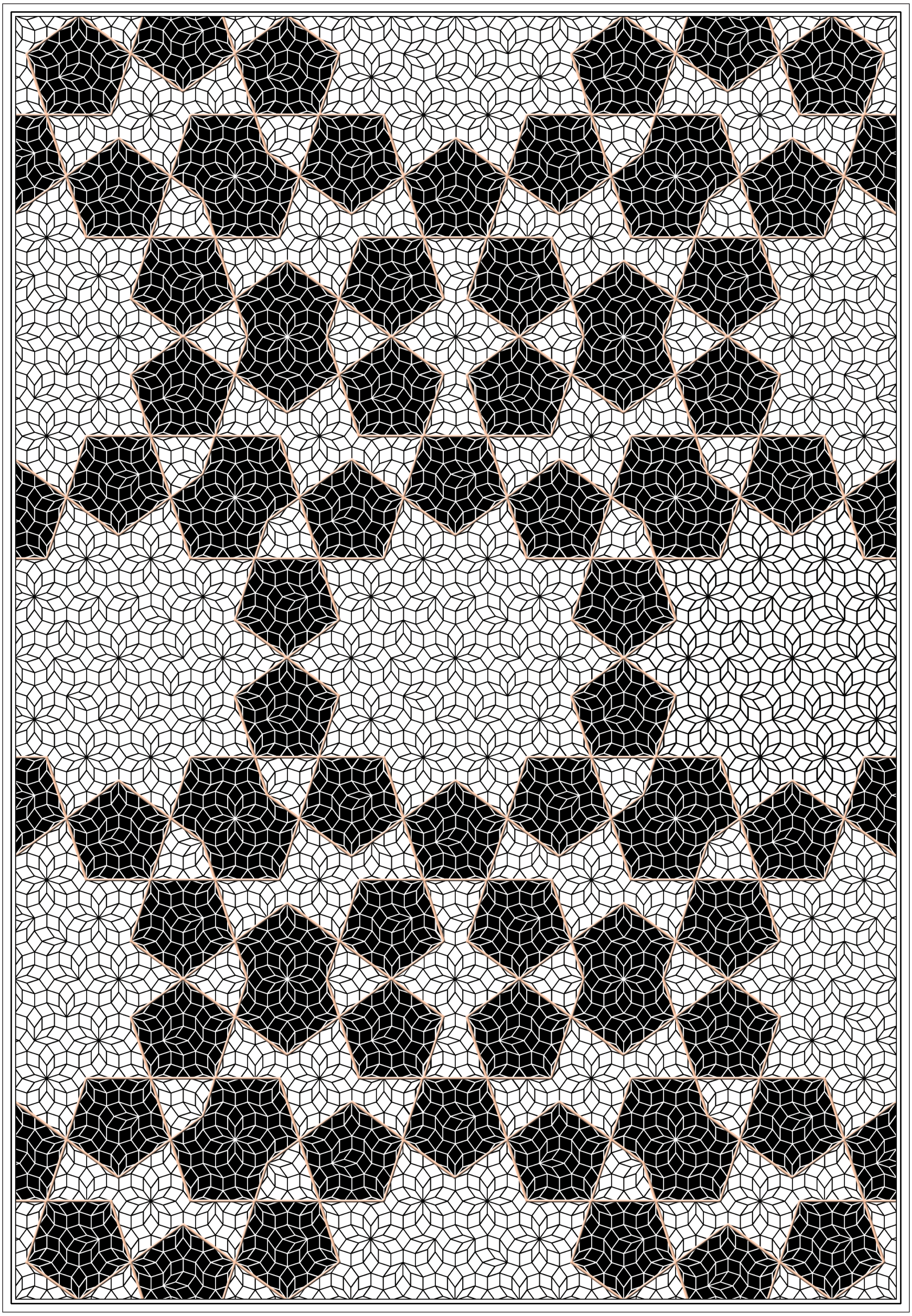


Figure 26. The second solution of mapping the tiles with Penta-Rhombs, applied on a pattern made of tiles from [S1].

6. Compatibility with the "X-Tiles", Binary Tiling and link with the Flower Family

We have seen that any pattern made from [S1] can give way to a pattern made of the two Penta-Rhombs. Is that pattern a Penrose Tiling? In fact, while mapping the tiles we were thinking about the "X-Tiles" previously discovered [3]. Though they are the same rhombs as in Penrose's patterns, they don't follow the same matching rules. The matching rules for the X-Tiles are the same as the ones in use for the "Binary Tiling" as defined in *quadibloc.com/math/pen02.htm*. They can generate non-periodic patterns and, unlike Penrose's matching rules, periodic patterns as well (so, the system tiles+rules is not aperiodic).
We are going to see that the two mappings proposed before are compatible with the X-Tiles. They generate patterns that belong to {F}, the Flower Family of pentagonal patterns.
A surprising result, isn't it?

Application of X-lines on the first mapping solution.

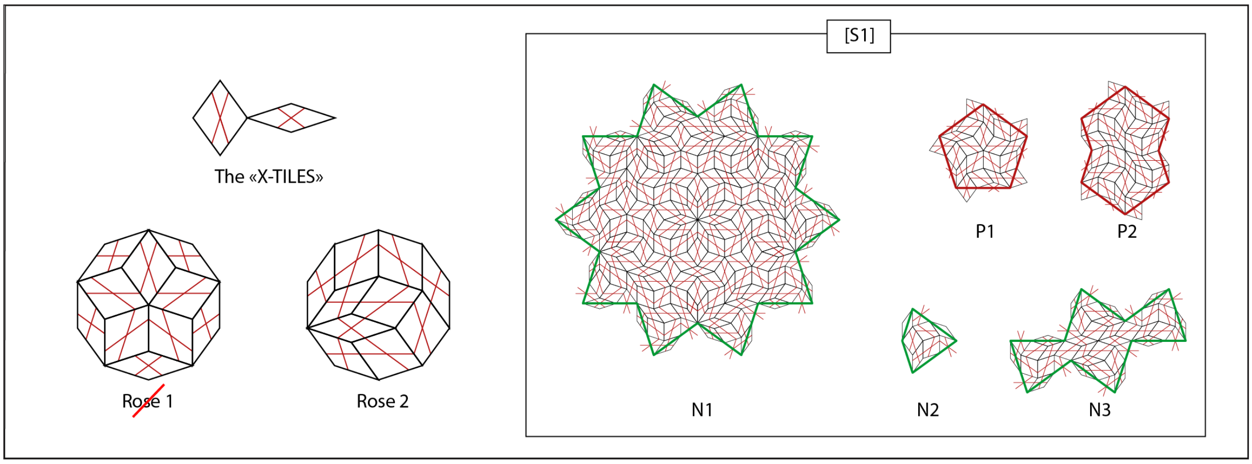


Figure 27. Left-up, the "X-Tiles", with the "X-lines" in light. Bottom, the two mappings of the Rose. Only the second is compatible with the matching rules for the X-Tiles (respecting the continuity of the X-Lines).
Right, we can see that the mapping of the set [S1] is compatible with the X-tiles.

Application of X-lines on the second mapping solution.

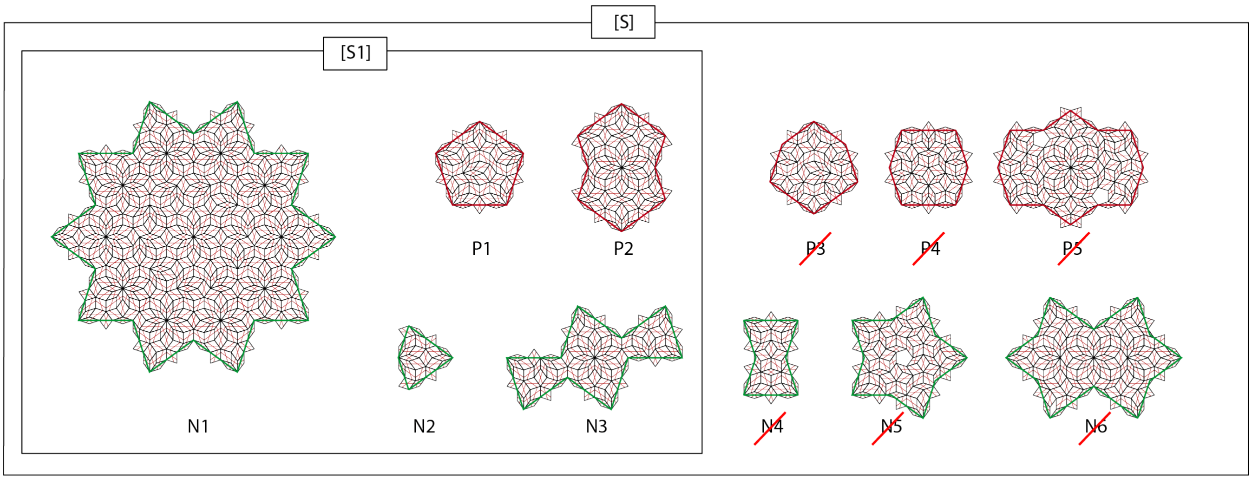


Figure 28. The X-Lines drawn on each Penta-Rhomb of the second solution of mapping shown in Figure24.

The whole set [S] is not compatible, but [S1] works perfectly.
Figure 29 shows the set [X] of all the tiles that can be drawn from the X-Lines. They belong to the traditional Iranian style "*Tond*" (Figure 30). As for the tiles of [S], we can distinguish Positives and Negatives tiles. Any pattern made with these 9 tiles (There are a lot) can be made from the X-Tiles.

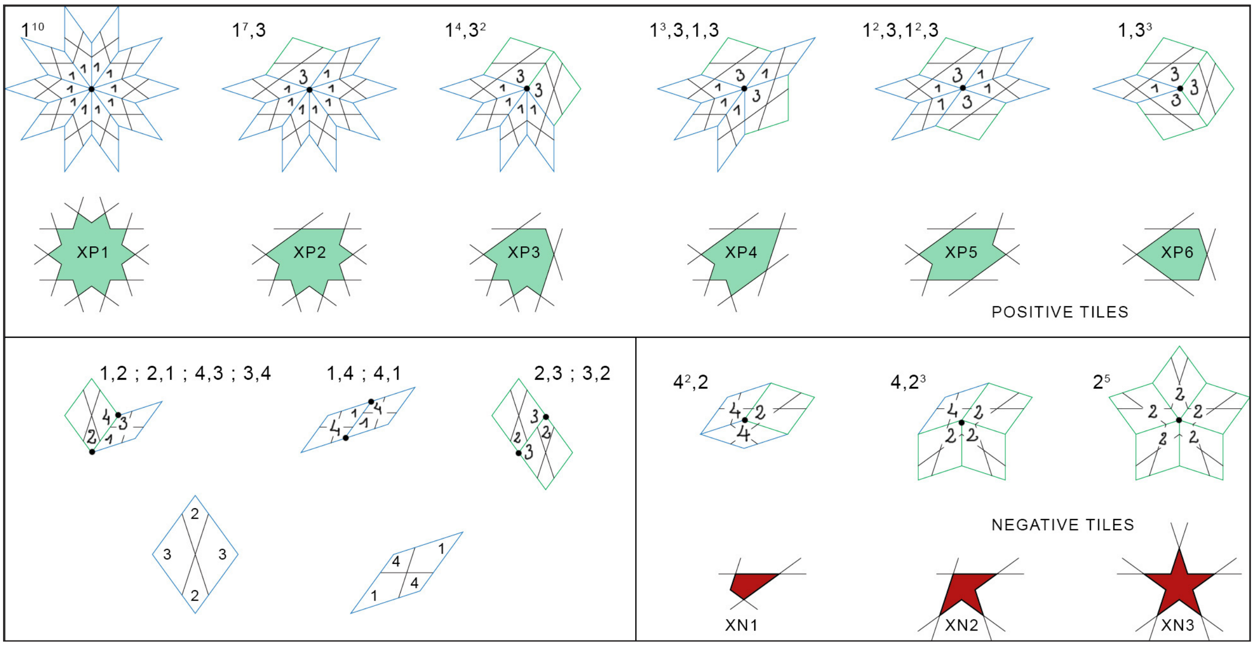


Figure 29. All possible arrangements of X-tiles around a vertex. In color, the associated tiles drawn by the X-Lines. The numbers 1, 2... 4 are notation for the angles (multiples of 36°). Bottom left, the impossible combinations. Image taken from [3].

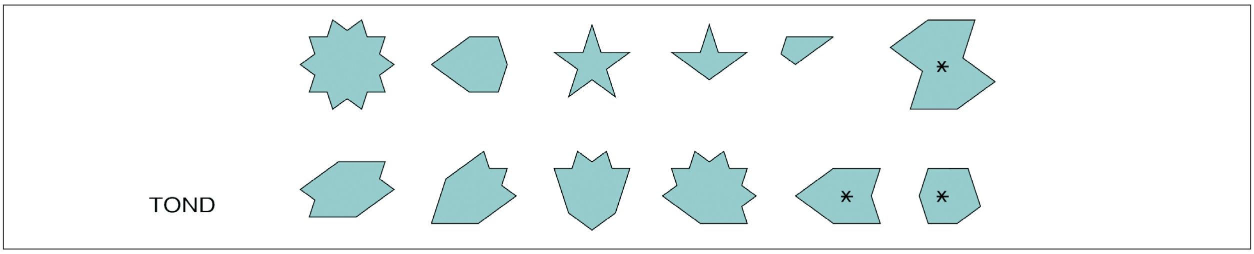


Figure 30. The Iranian traditional set of tiles "*Tond*" include 3 extra tiles (X) compared to the set of tiles made from the X-tiles. Note that in this set the {10/3} star is undecorated.

Next pages, some illustrations.
Again, we are starting from the same pattern. The figures 31 and 33 comes from the first mapping system after replacing each rhomb by the associated "X-lines", and from the second on figures 32 and 34.
On the two last figures we have highlighted the tiles along the edges of the original pattern. You can see that it works perfectly with the second mapping. Even though it looks similar to the skeletons in use for the octagonal system in Morocco [5], it does not follow the same logic.

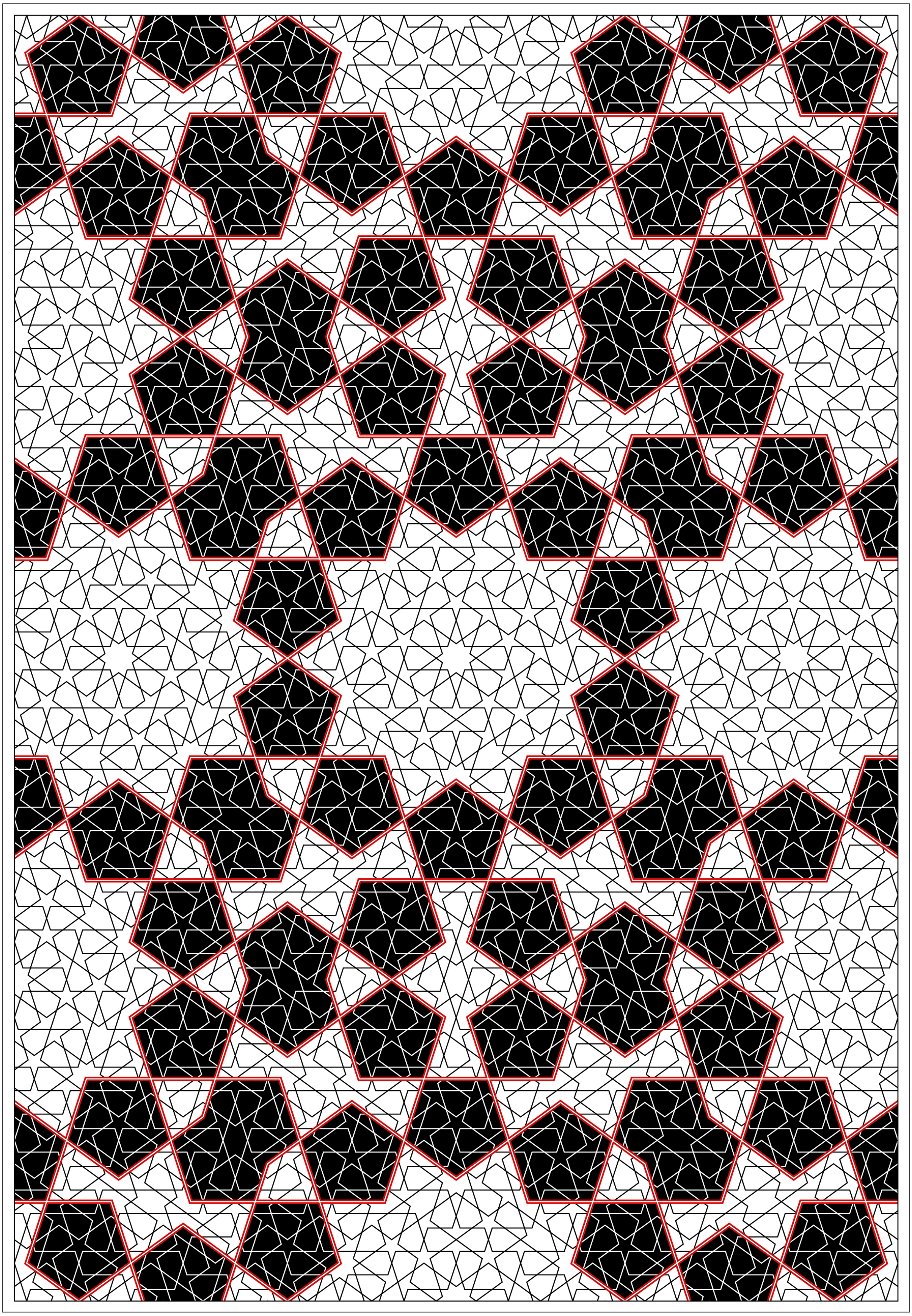


Figure 31. After drawing the X-Lines on the rhombs of Figure 25 we got a pattern of the family {F} at the second level.

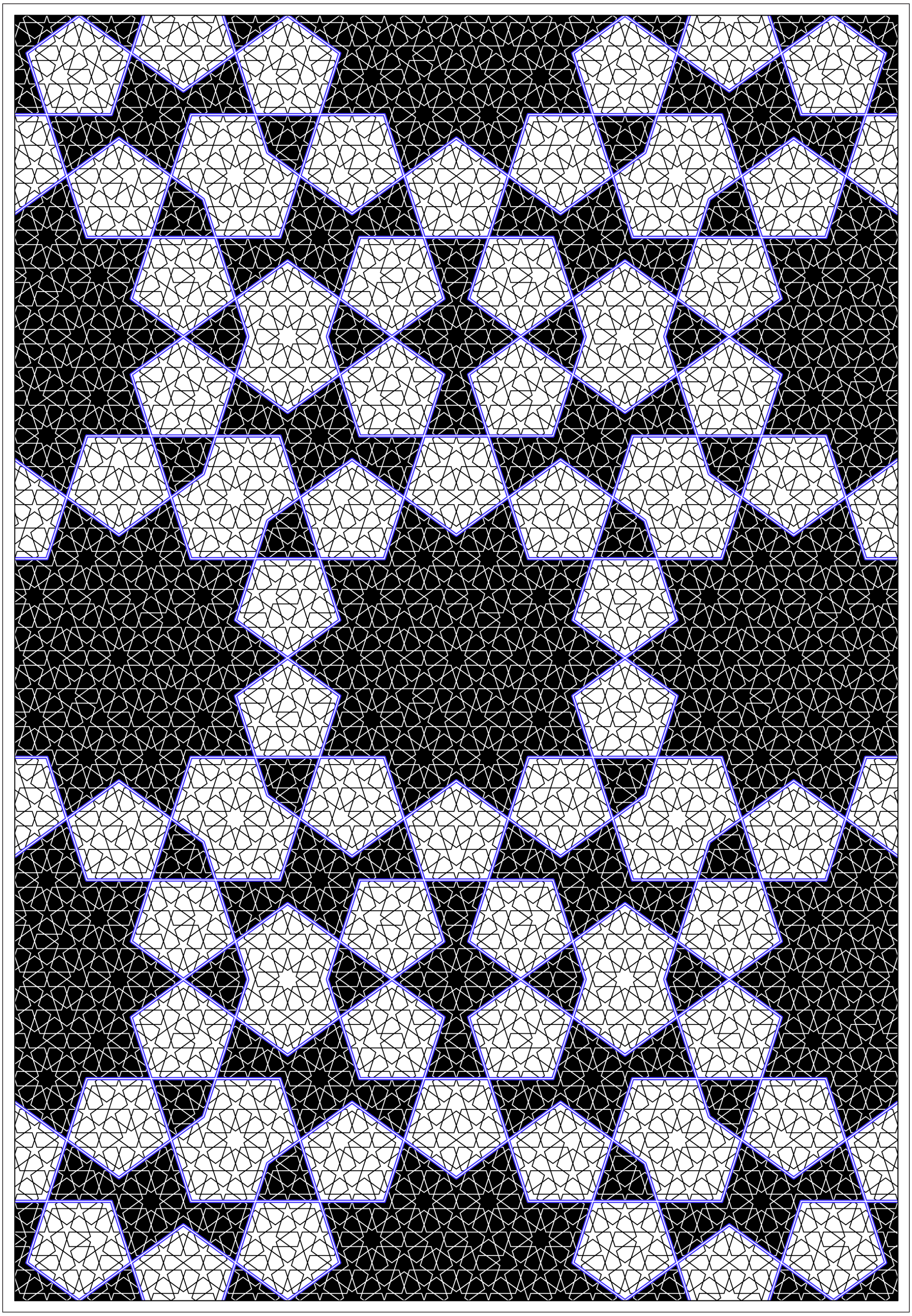


Figure 32. Same transformation applied on Figure 26. The second level pattern belong to the Iranian style “*Tond*” .

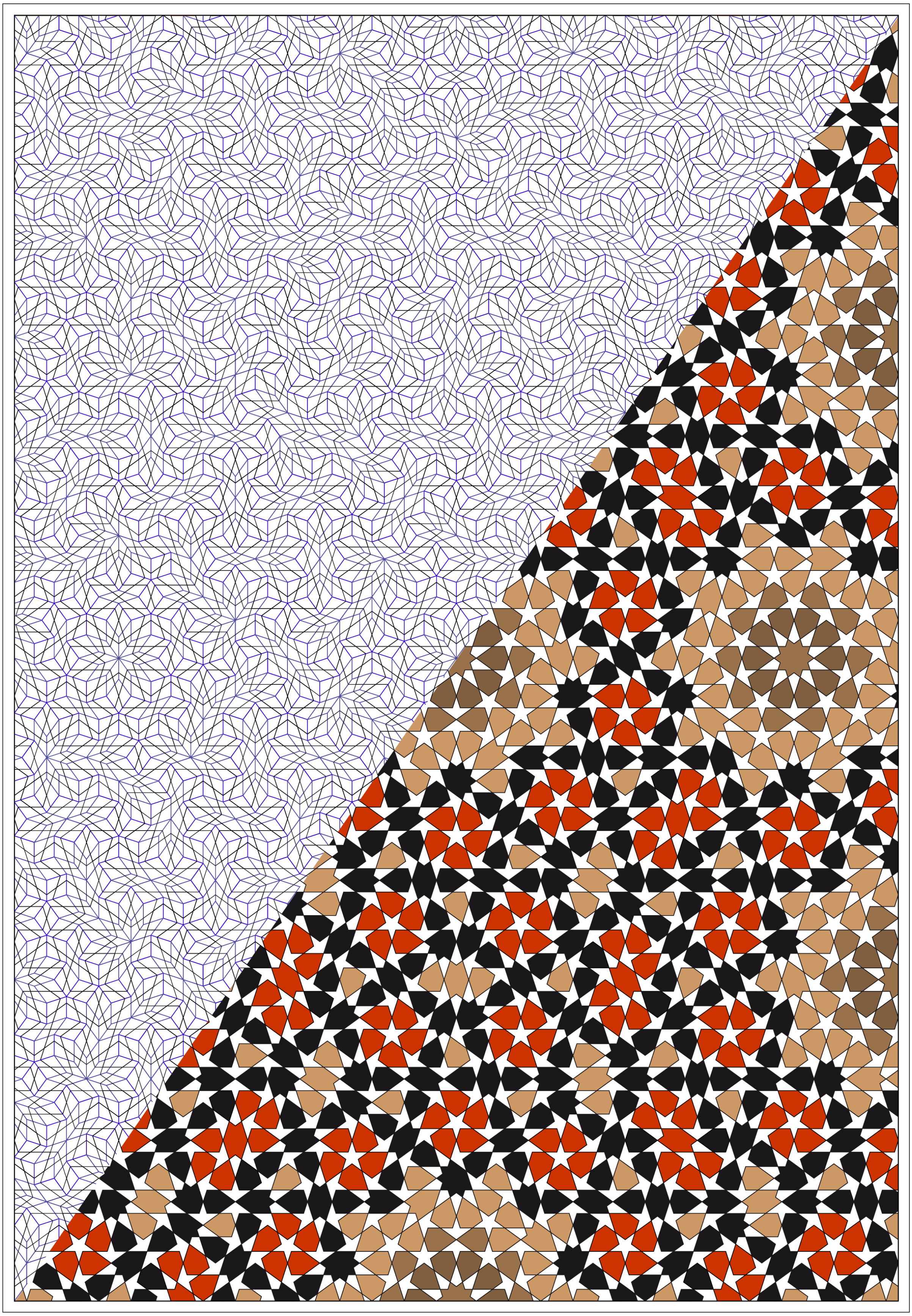


Figure 33. Second level pattern of the figure 31.
We have colored in black all the Positive Tiles crossing the edges of the first level.

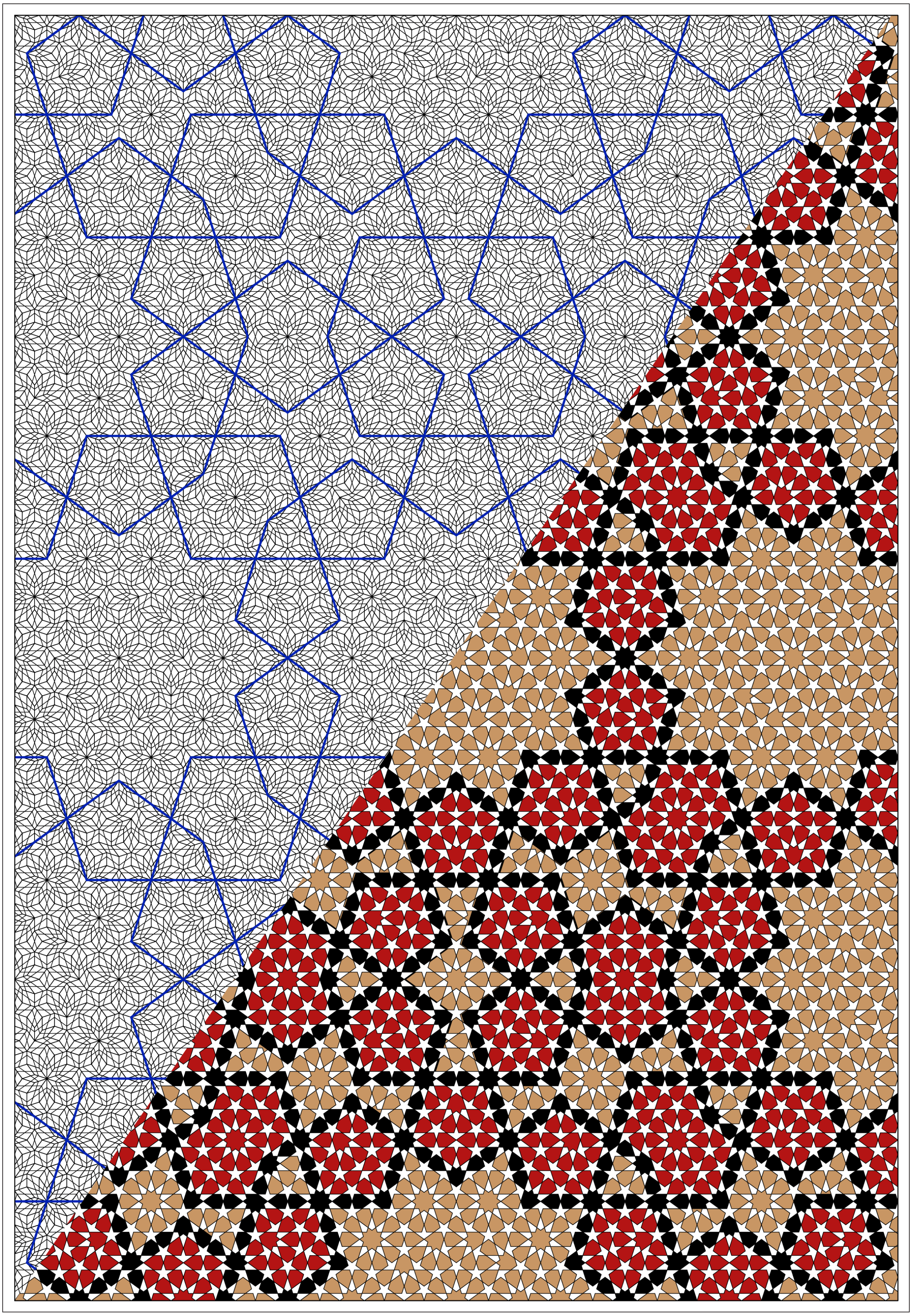


Figure 34 . Same style of coloration applied to the figure 32

7. Examples in the Iranian traditional Architecture

7.1. Starry family {S}. Variations on the "mother of tilings" K1, the simplest *Kond* pattern.

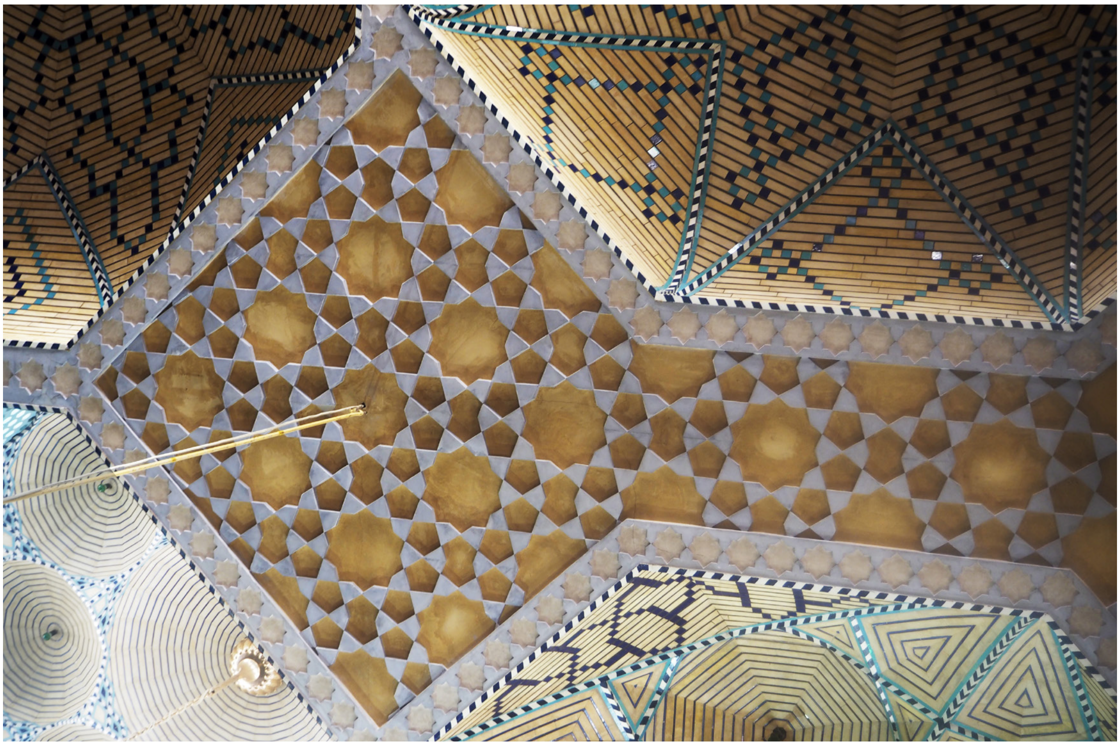


Figure 35. "The mother of tillings" can be considered here as made from only one kind of shape, the "*Toranj*" (N2). Part of the roof of a new mosque in Isfahan.

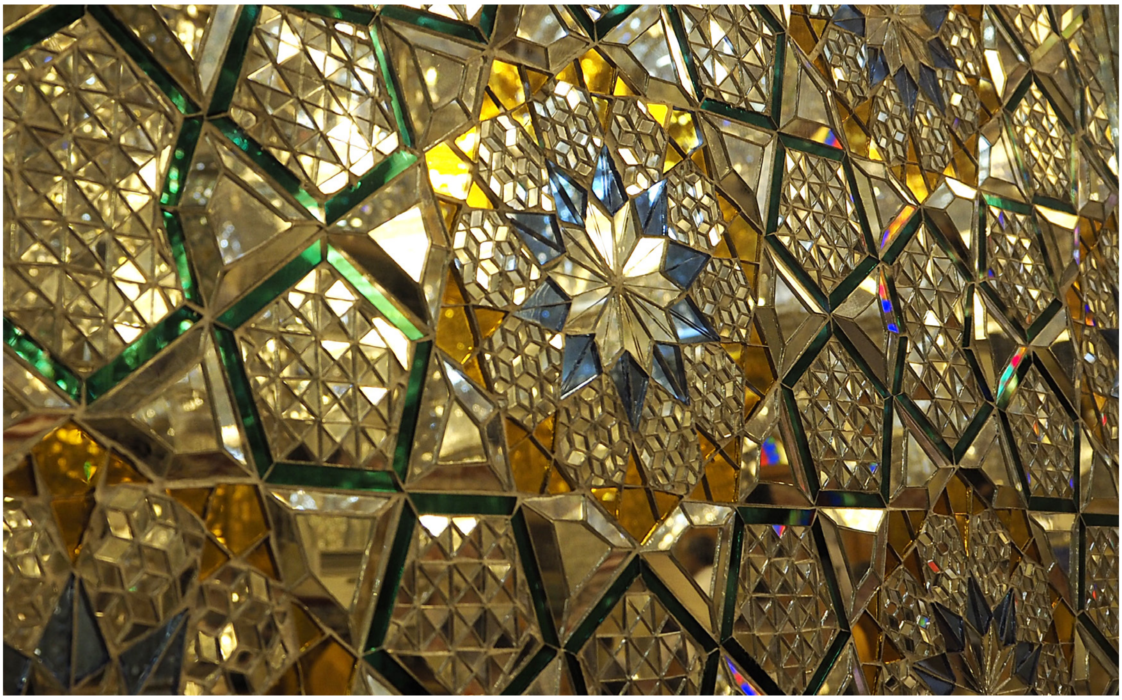


Figure 36. The same pattern made of mirrors in Shiraz, Shah Cheragh shrine. With insertion of a rosette into the central star.

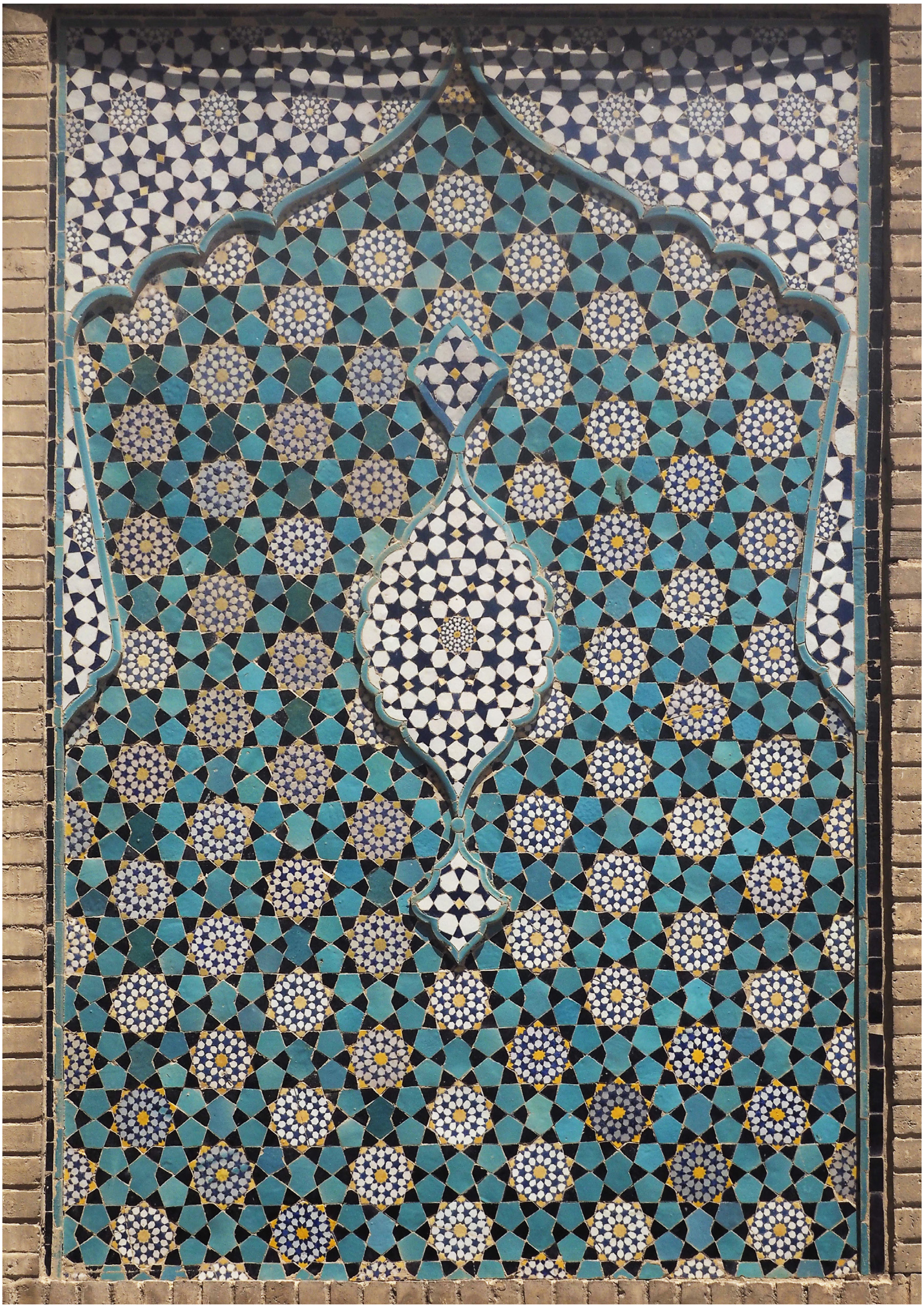


Figure 37. Isfahan, Friday mosque. Same simplest *Kond* pattern, K1, with variation (type 1) of the *Shamseh*. The pattern at the background, the central medallion and the decoration of the *Shamseh* belong to the *Kond* and *Shol* Iranian family. According to this paper, every tile belong to the set [S] (with variations of N1 and N6), at different scales.

7.2. Flower family {F]. Variations on the simplest *Tond* pattern, T1.

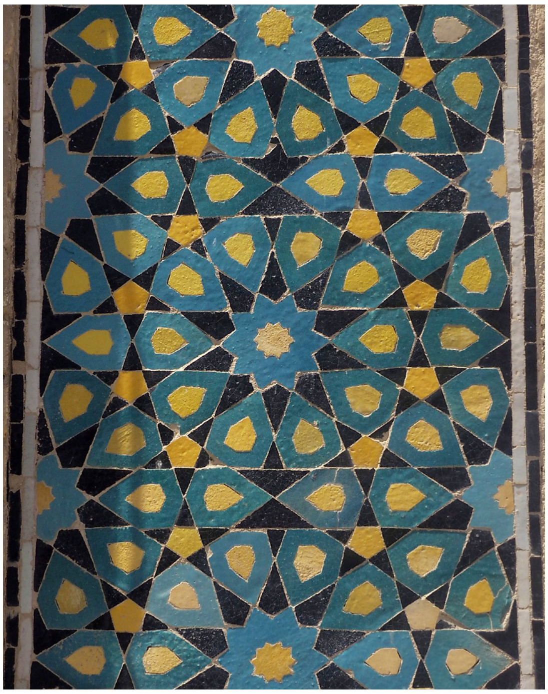


Figure 38. Isfahan, Friday mosque.

Figure 39. Isfahan, Hasht Behesht (The eight paradises).

Figure 40. A door in the old bazar of Isfahan.

Figure 41. Kachan, a door at Agha Bozorg shrine.

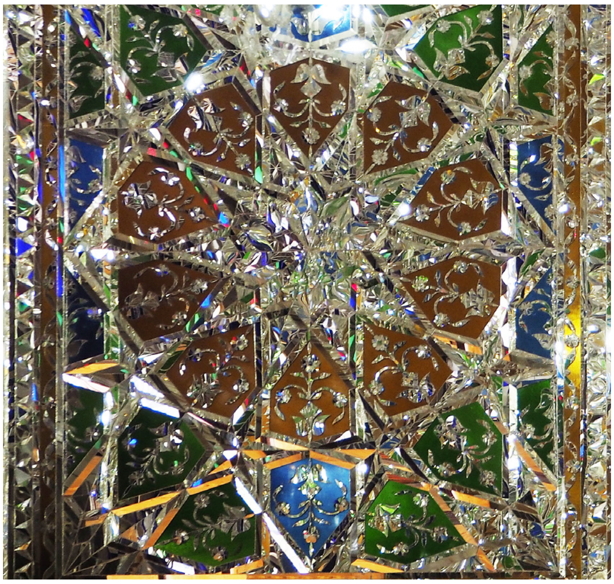


Figure 42. Yazd, mirrors in a shrine.

Figure 43. Shiraz, in the old city.

A same pattern may look very different according to the material and the techniques used, this constitutes also a kind of variation. The simplest *Tond* pattern can be seen made of mosaic of ceramic (Figure 38), simply engraved on a wall (Figure 39), in wood (Figure 40, 41), in mirrors (Figure 42), sometime with slight variations (Figure 40). It can be decorated in a way that makes a link with the *Kond* family (Figure 43), as explained below.

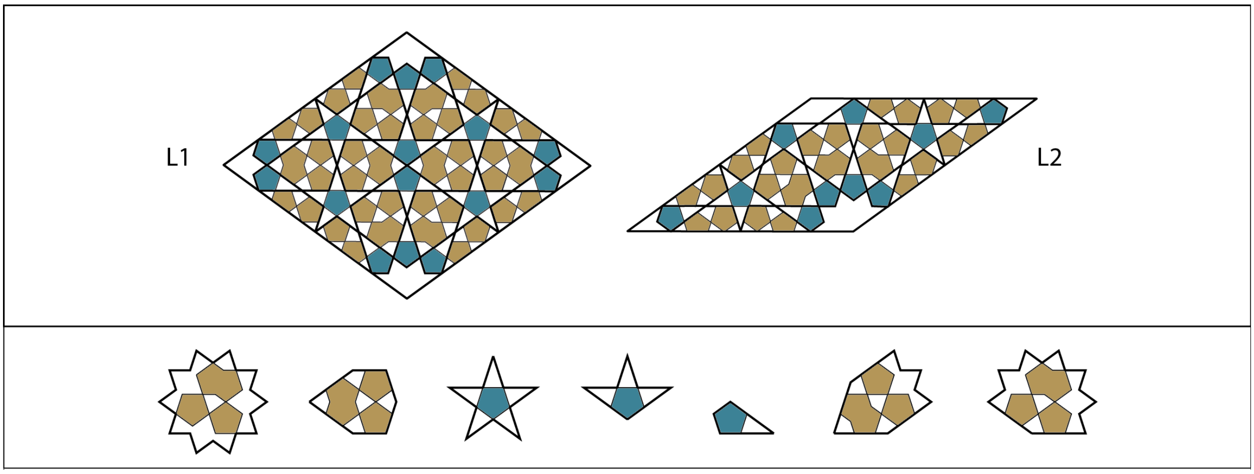


Figure 44. A process of transition between the *Tond* (bold lines) and *Kond* (thin lines) patterns.

The rhomb L1 on Figure 44 is the unit cell of the standard pattern T1 with each *Tond* tile decorated according to the rules shown at the bottom of the figure. The rhomb L2 is the complementary shape (see [3]). The mapping of each edge of the rhombs is identical and symmetric, so this decoration of the two rhombs can be applied to any Binary tiling and any Penrose tiling as well.

The tiling on Figure 43 is not a perfect implementation of this process, because the decoration of some tiles uses another scale.

Note: This process does not works automatically for any *Tond* or *Kond* pattern.

7.3. Strange Patterns.

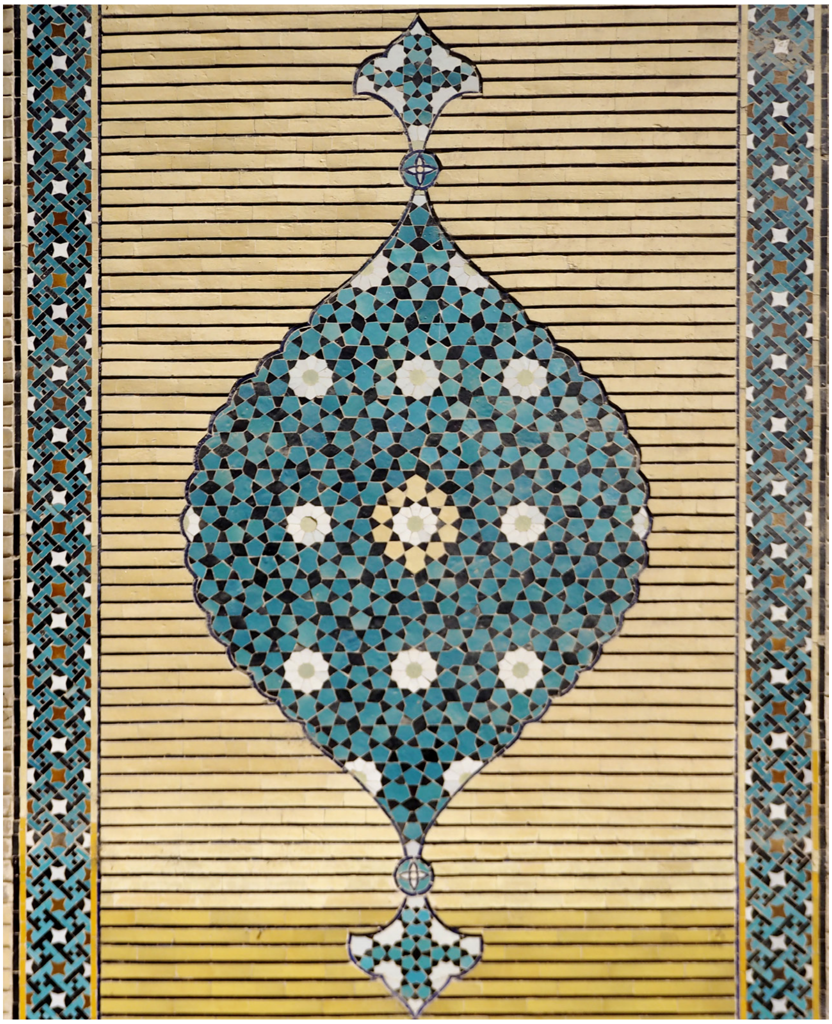


Figure 45. A medallon in mosaic of ceramic at Chahar Bagh Madrasa, Isfahan.

The patterns are not only used to fill rectangular surfaces. Sometimes they have to adapt to a curvilinear shape, as for the medallion in Figure 45. This *Kond* pattern is more complex than the simple "mother of tilings" K1 which we have seen before. At first sight it seems to be made of the repetition of a rectangle with a *Shamseh* at each corner (Figure 46-a, 4 of such rectangles). But after simplification by removing all variations of N1 (Figure 46-b, c and d), it appears to be basically nothing but the simple K1 pattern with variations $N1_1$ and $N1_2$ (Figure 46-d, the pattern is shown as repetition of a lozenge).
However, compared to K1 some areas are slightly different (Figure 46-e), and the "hidden" tiles N1 are decorated in a way that does not seems to follow an obvious logical rule (Figure 46-f).
We have chosen to highlight the orientation of the $N1_2$ variation with an arrow (Figure 47).
I certainly do not consider this lack of symmetry as an error. In my opinion, this adds life to a pattern which is inserted into a vegetal shape.

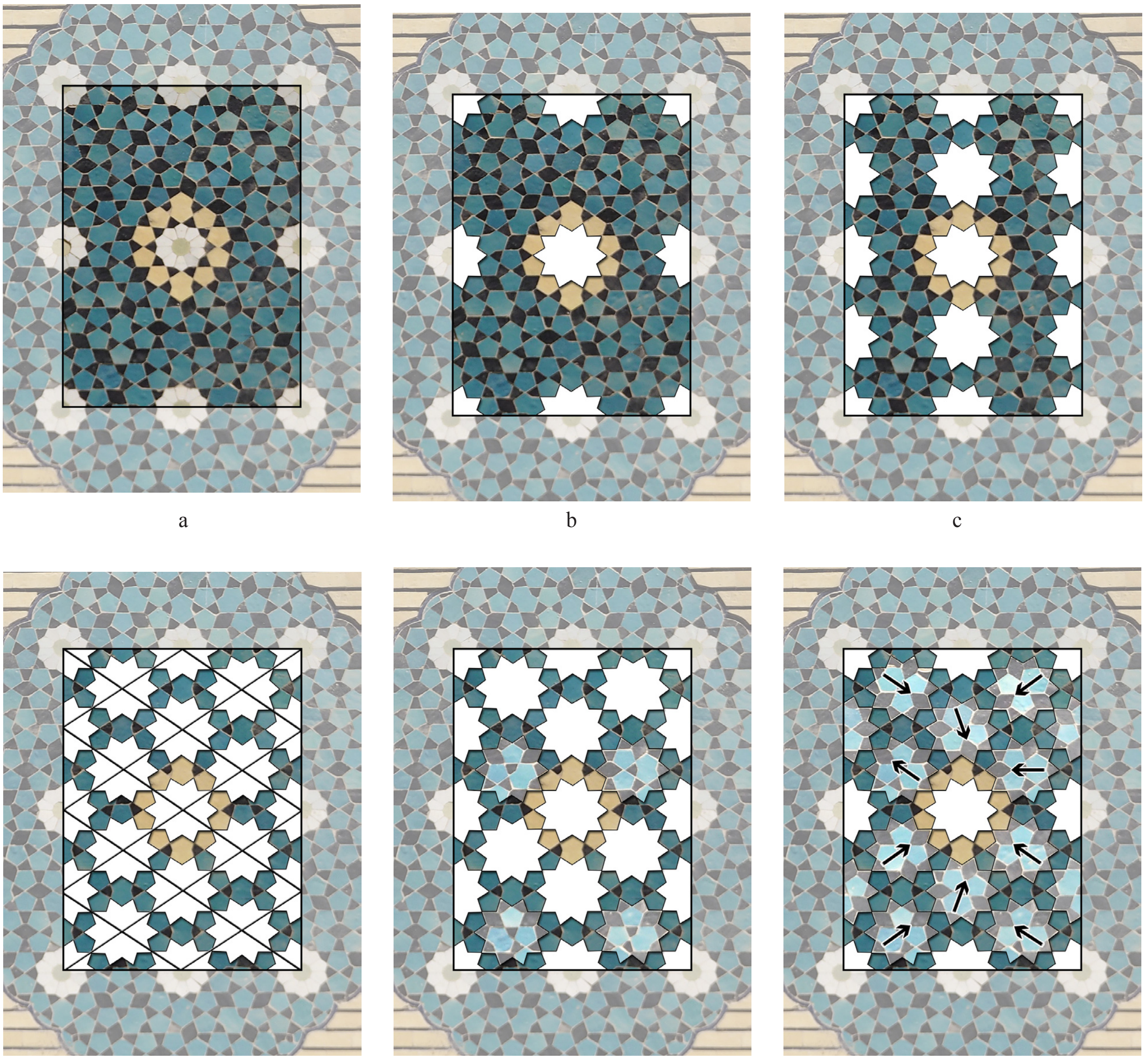
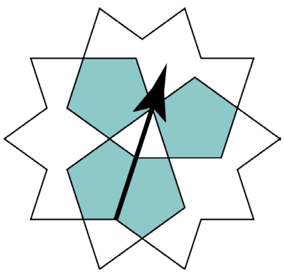


Figure 46. The pattern of the medaillon, considered as a variation of the “mother of tilings”.



igure 47. Orientation convention for the variation $N1_2$ used in 46-f.

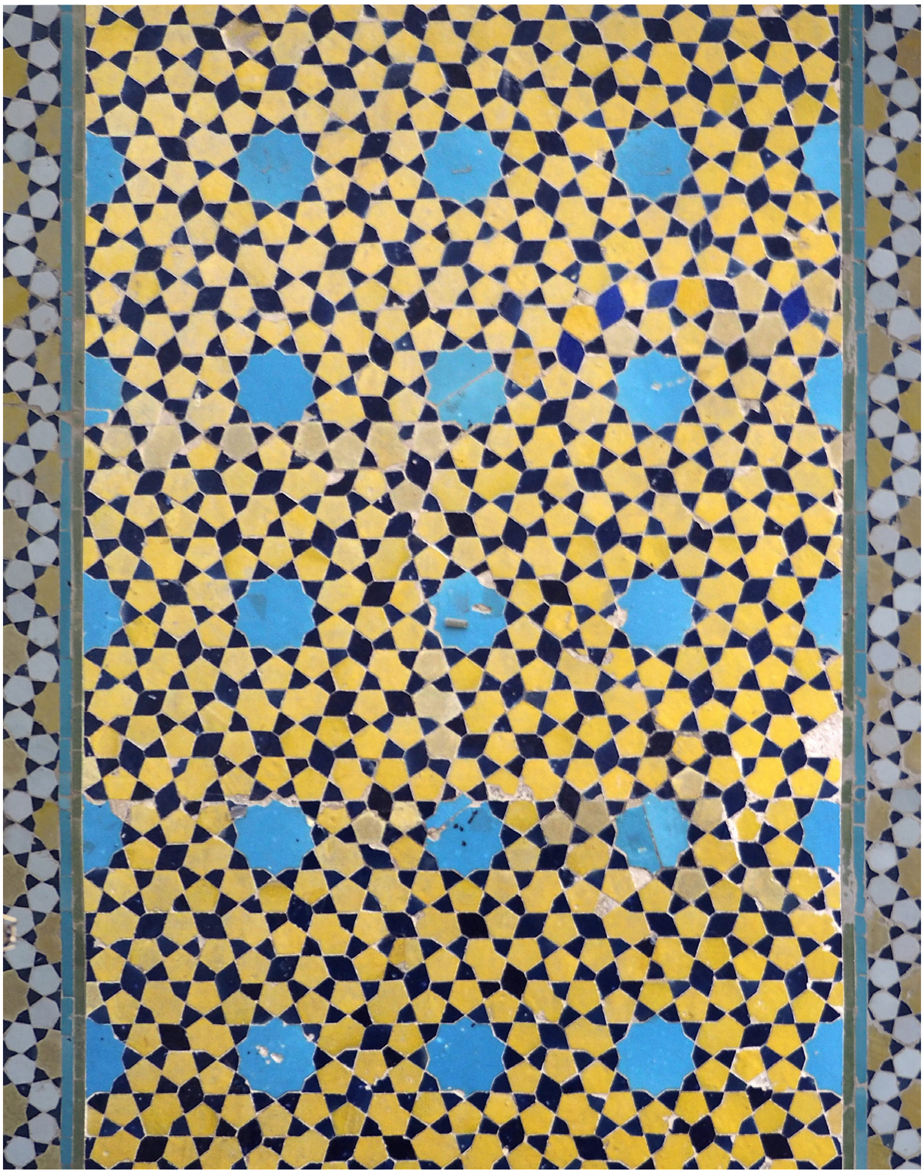


Figure 48. Friday mosque, Isfahan.

In the same style, the pattern on Figure 48, at the Friday mosque in Isfahan, is particularly remarkable. Again, at a first sight it can be seen as repetition of a rectangle (Figure 49-a). At the next step of simplification we can recognize the regular repetition of two overlapping stars (Figure 49-b). But the decoration of these shapes is not regular. In Figure 49-c we have used the same convention as before (Figure 47) to show the orientations. Same thing in between, the arrangement of the tiles is not regular as shown with the black arrows (Figure 49-d).
Again, there is certainly no error in this non-periodic tilling. This gives the viewer a strange feeling made of the contrast between the strict order of the main stars and the disorder of other tiles, sparkling like reflections on the water, or leaves in the wind. Could it be a coded message? Why not, I guess Muslims had the sense of humour in the old times. But I'm not the one who will even try to decode it.
This tiling would be difficult to analyze without using any simplification.

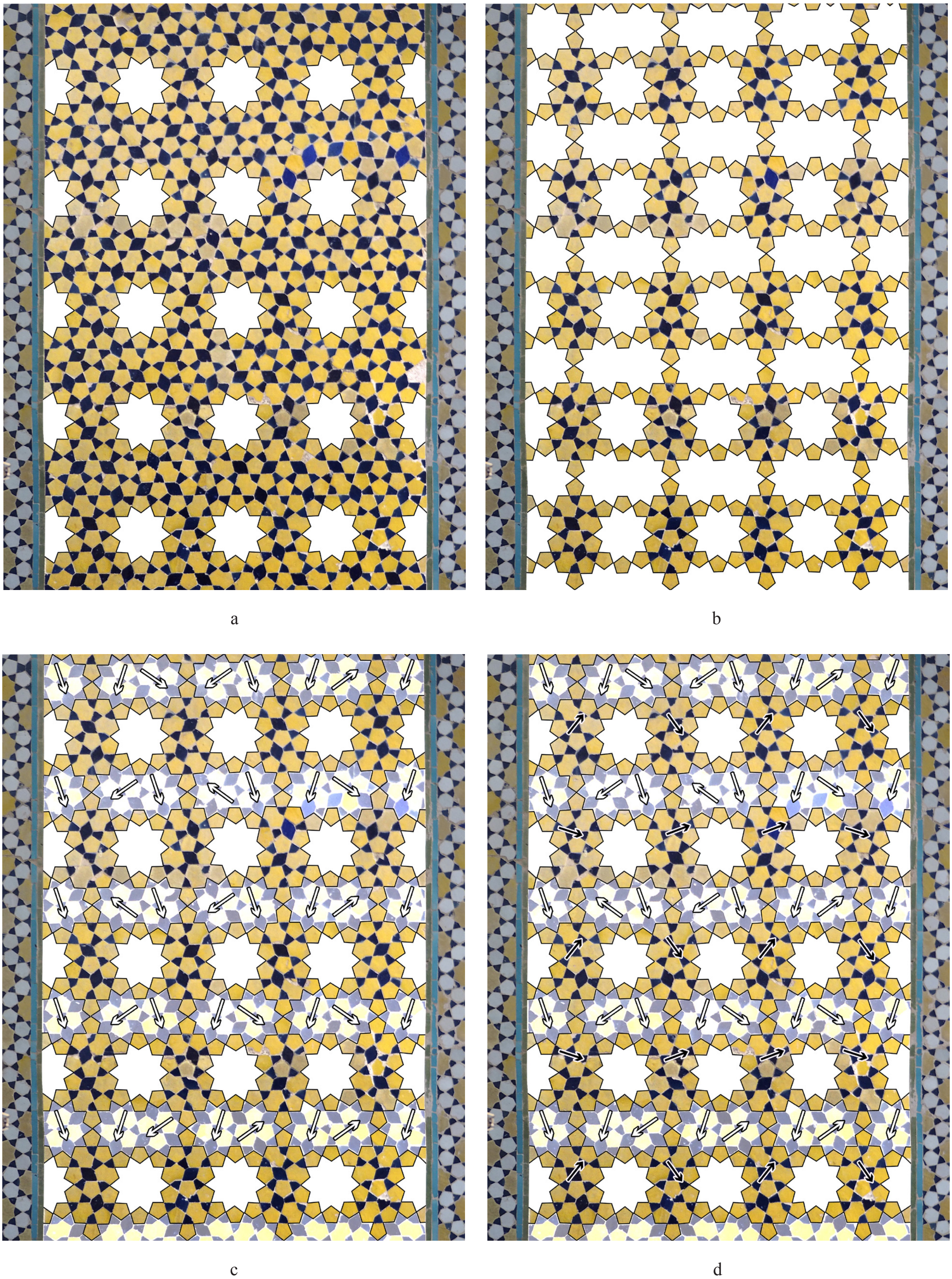


Figure 49. (a): Periodic repetition of the stars. (b): Overlapping stars in between.
(c): Non periodic orientation of the decoration of the overlapping stars and... (d): of the tiles in between.

Figure 50. Entrance Iwan of the Friday mosque, Isfahan

An other strange pattern can be seen on the spandrel of the main entrance of the Friday mosque in Isfahan. All tiles belongs to the set [S1]. In this case, I cannot find any efficient simplification or other process that could reveal a hidden regular structure.
Maybe a reader of this paper can make it. In my opinion, the artist started with the idea of a *Kond* pattern in which he could highlight a cursive drawing by the use of color. This cursive drawing could be considered the first level pattern.

7.4. The X-Tiles in historic patterns.
The two wooden doors in (Figure 51), although they are not from Iran but from the Serefeli mosque, Edirne, Turkey, belongs to the Persian style. The first pattern (Figure 51-a) is the simple T1, while the second (Figure 51-d) is a variation. On the images at the middle we have superimposed the Penta-Rhombs, so you can see the X-lines of the pattern inside. On the left, the reconstruction of the pattern with the tiles of the "X-Puzzle", a game I have specially made.
I do not pretend that the historic artists had used this process.
I wonder if it worth it to search a Penrose pattern in a tiling which can be reduced to a Binary tiling…

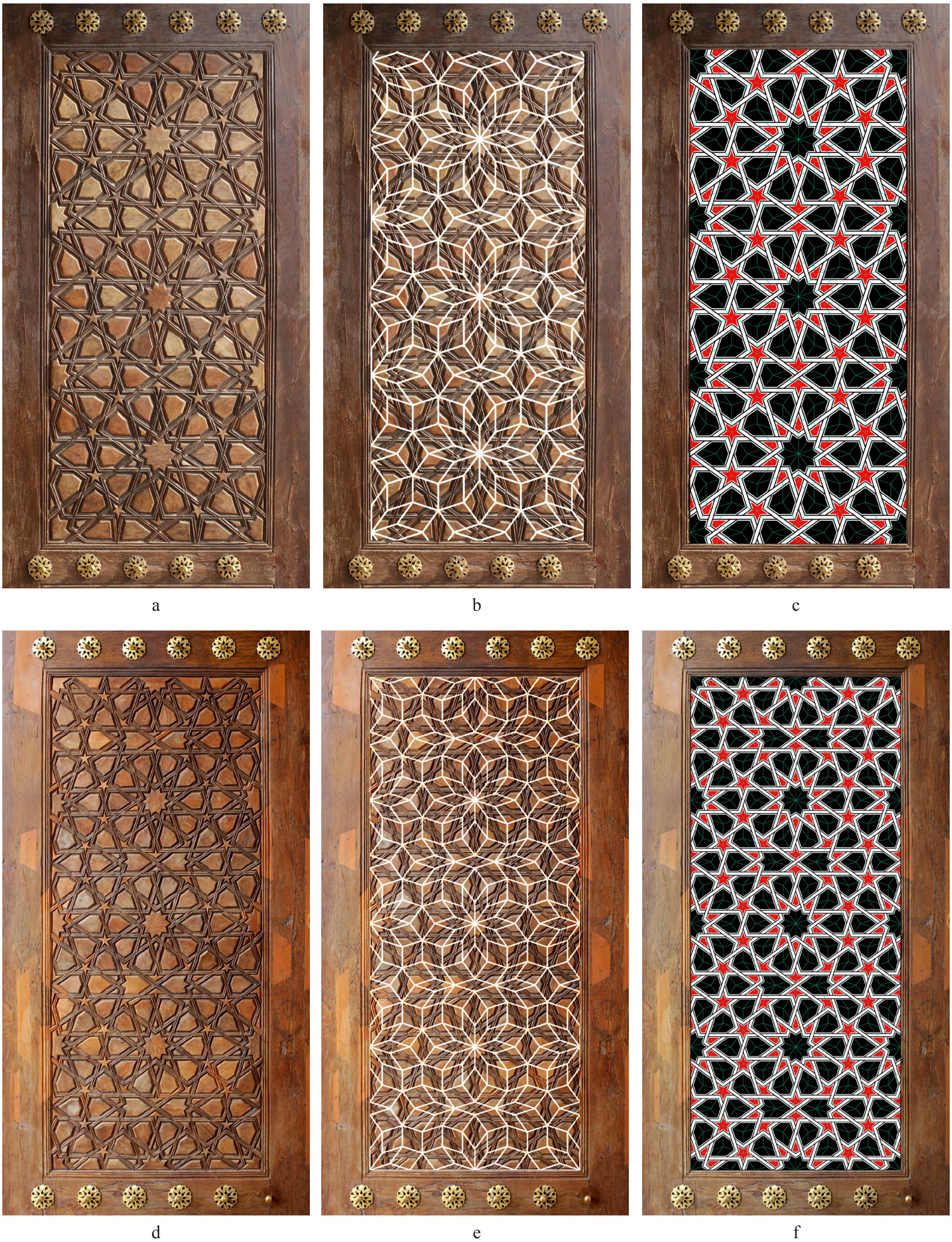


Figure 51. Two doors at Serefeli mosque, Edirne, Turkey. Analysis and reconstruction in terms of X-tiles.

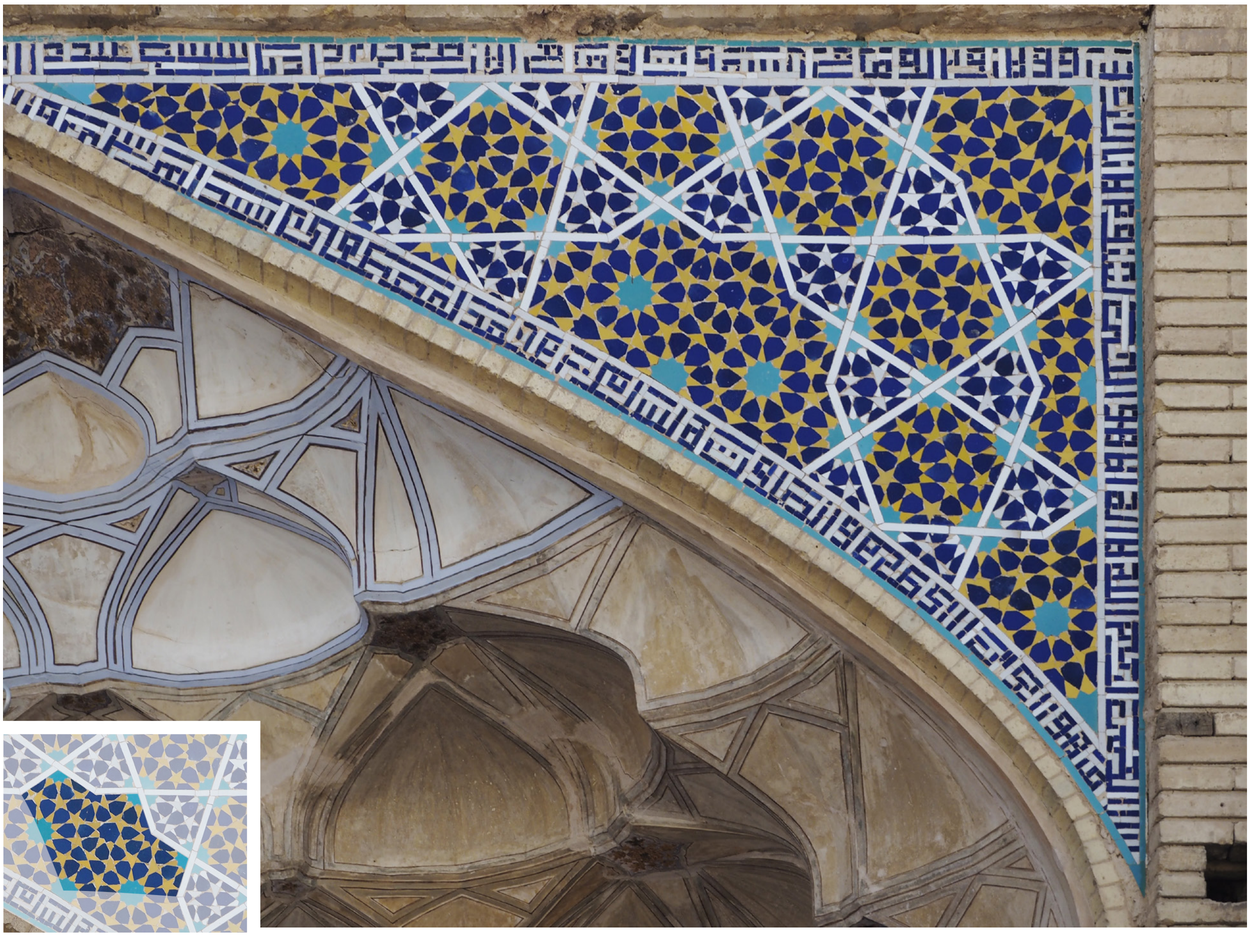


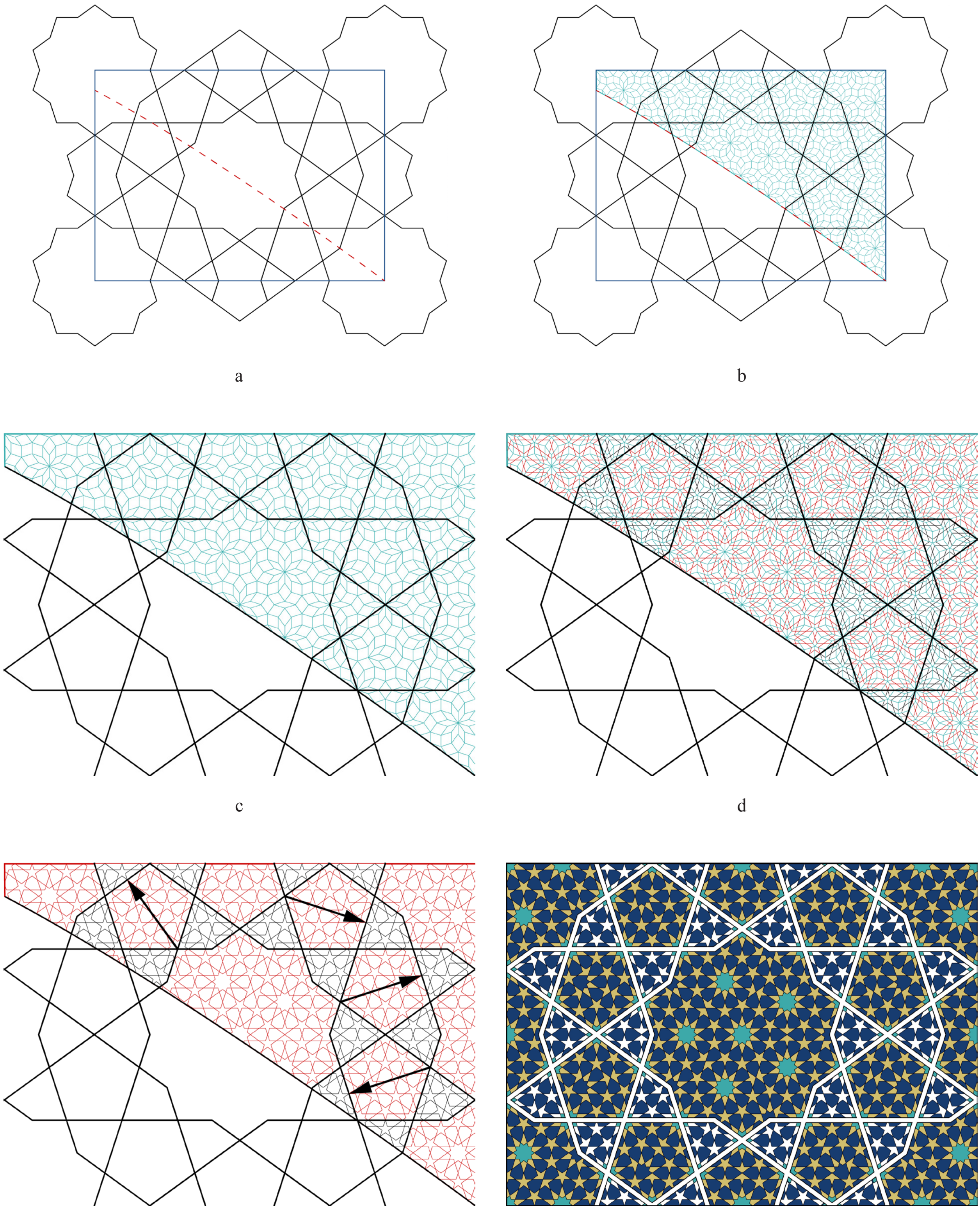
Figure 52. A spandrel at Darb-e Imam, Isfahan.

This pattern on a spandrel in Darb-e Imam, Isfahan (Figure 52), is made of two level. The first level is a *Kond* tiling, again the simplest K1, while the second belongs to the *Tond* family. The filling of the central star has been adapted to the shape of the spandrel. A quick sight at the tiles is enough to understand that this pattern could be made of our X-Tiles: indeed, they all belong to the set [X] (Figure 29).
Figure 53-a shows the basic pattern K1. In the Figures 53-b and c we have drawn the Penta-Rhombs, from which the pattern can be reconstructed when we add the X-Lines to each rhomb (Figure 52-d). Notice that we have filled the Shamseh in a regular way, contrary to what we see in the actual tiling.
On the Figure 53-e we have removed the rhombs and added arrows to show the irregular orientation of the pentagons.
Figure 52-f shows the whole pattern in a periodic version, inserted into a rectangle, with regular orientation of the pentagons and Shamseh. This pattern constitute a slight variation to the solution given in chapter 5 (Figure 23, 28, 32): The mapping of the tiles P2, N1 and N2 are slightly different.

In term of self-similar pattern, the characteristics of this one are:

Styles : From *Kond* to *Tond* (First level *Kond* style, second level *Tond* style)
1st level: K1 pattern, periodic. Tiles: P1, P2, $N1_1$, N2
2nd level: All tiles belongs to [X].
This is a 2-level pattern, non self-similar because the tiles of the two levels do not belong to the same set.

We are going to see more multilevel patterns in the next section.



a b c d e f

Figure 53. Analysis and reconstruction with the X-Tiles.

7.5. Multilevel Patterns.

These examples are sometime call self-similar [1], but they are not in the strict sense defined in chapter 4. They are only 2-level patterns, although some of them could be easily transformed to be self-similar.

Convention: We call "System 1" the first system of Self-Similarity defined in chapter 4 (Figure 15), and "System 2" the second (Figure 17).

Example 1, Figure 54. A spandrel at Shah Cheragh, Shiraz.

Characteristics:

Styles: From *Kond* to *Kond*.
Inflation rules: System 1.
1st level: K1 pattern, periodic. Tiles: P1, P2, $N1_1$, N2
2nd level: All [S1] tiles: P1, P2, $N1_1$, N2, N3.
Second option of inflation for the tile P1 (P1b on Figure 15).
Two-level pattern, globally periodic. Not Self-Similar because the inflation rule is not define for the tile N3 (Sormedan), and the first level, periodic, cannot be mapped onto the second one.
Particularities: The mapped pentagons P1b are laid out always along the vertical axis of the wall (Figure 55-b).This is the artist's choice.
Can we define a Self-similar pattern from it, with some modifications? Yes: System 1 with a slight variation for N1, and the use of the mapping P1b.

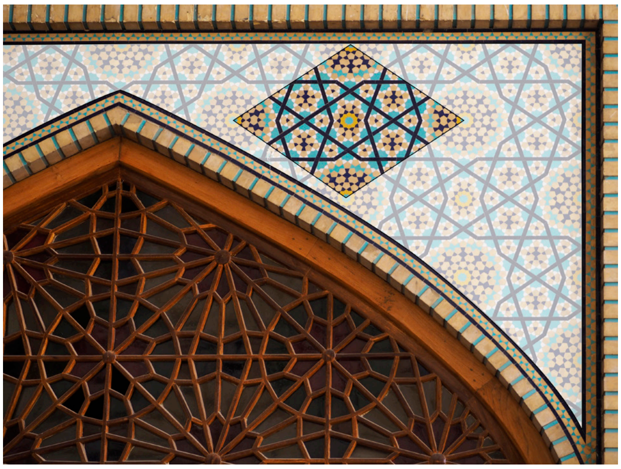
Example 2, Figure 56. Underside of an arch at the Vakil mosque in Shiraz.

It is the same as the previous one with slight differences in how the mapped P1b are laid out.
We have drew arrows to show the orientations of the pentagons, and highlighted the variations $P1_2$ at the centre, to show the differences with the option on chapter 4, Figure 15.

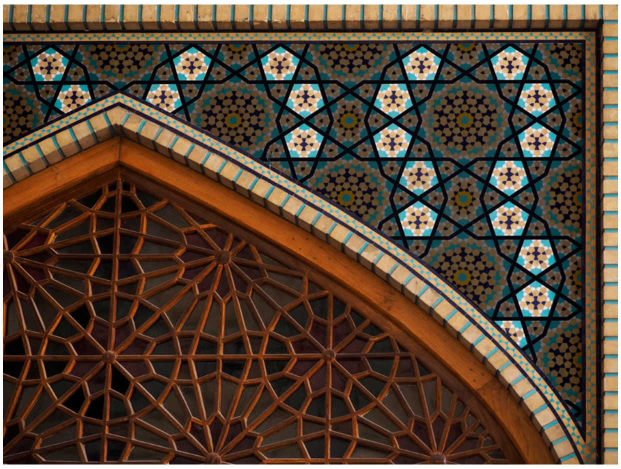
Characteristics:

Styles: From *Kond* to *Kond*.
Inflation rules: System 1.
1st level: K1 pattern, periodic. Tiles: P1, P2, $N1_1$, N2.
2nd level: All [S1] tiles: P1, P2, $N1_1$, N2, N3.
Second option of inflation for the tile P1 (P1b on Figure 15).
Two-level pattern. Not Self-Similar because the inflation rule is not define for the tile N3 (*Sormedan*), and the first level, periodic, cannot be mapped onto the second one.
Particularities: The mapped pentagons P1b are laid out in a puzzling way (Figure 56, right), which makes this pattern non periodic.
Can we define a Self-similar pattern from it, with some modifications? Yes: System 1, with a slight variation for N1, and the use of the mapping P1b.

Figure 54. Example 1. Shiraz, Shah Cheragh.

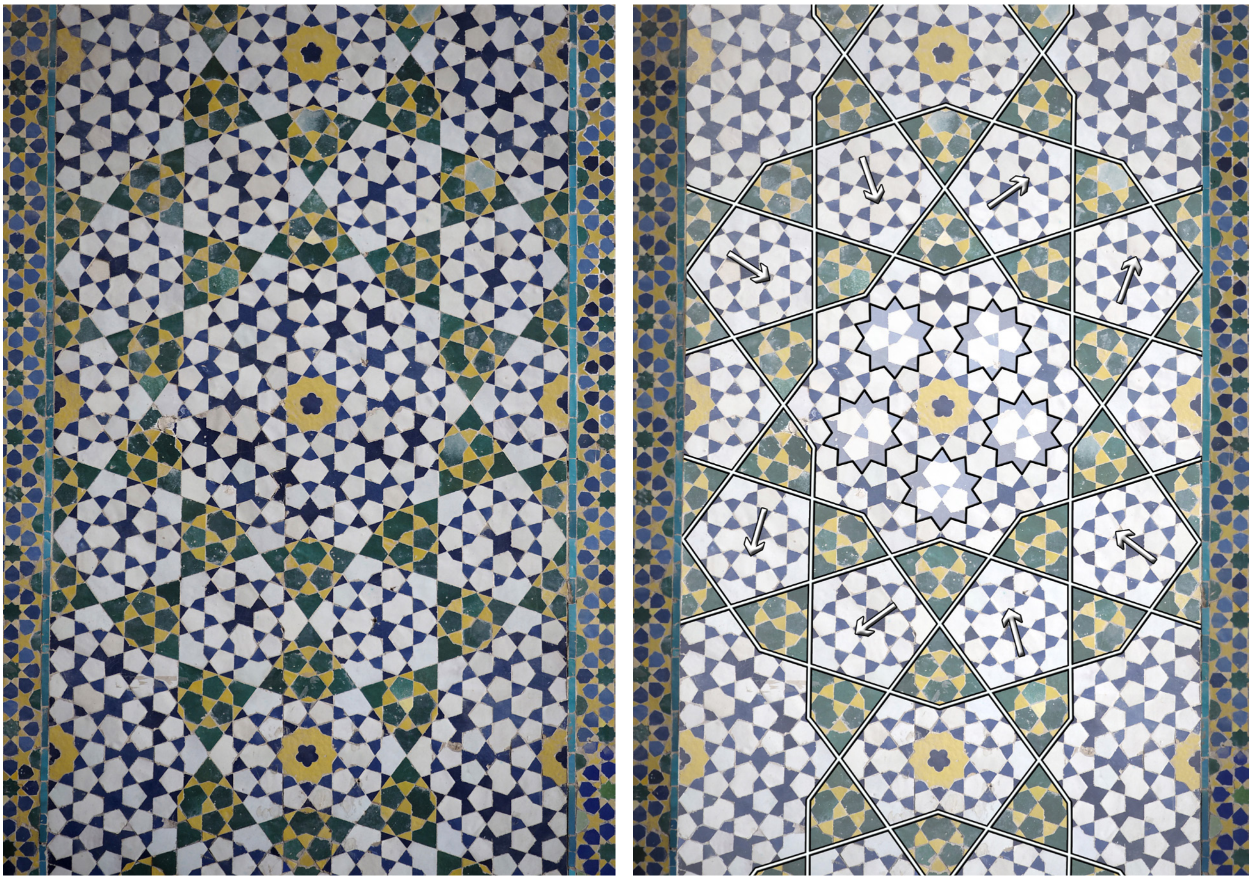


a



b

Figure 55. (a): The unit cell of the “Mother of Tiles” K1. (b): Vertical orientation of the pentagons.



a b

Figure 56. Shiraz, Vakil Mosque, part of the underside of an arch. Right: Orientation of the pentagons, and stars at the centre to be compared to Figure 15, chapter 4.

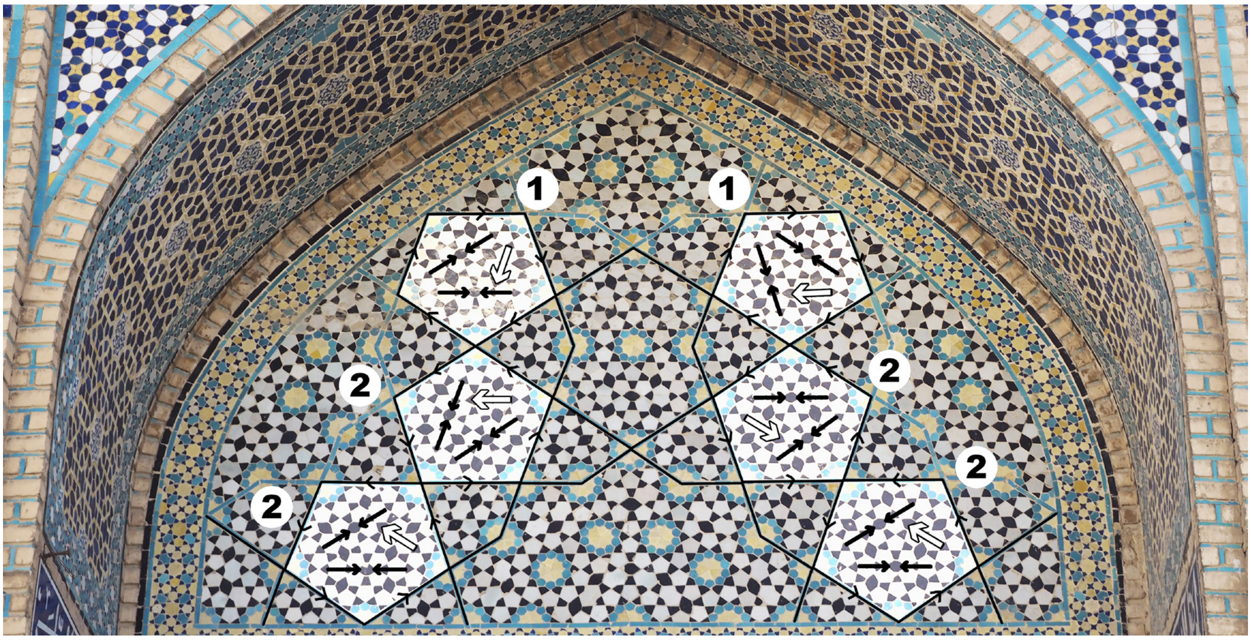
Example 3, Fig 57: The famous Darb-e Imam pattern in Isfahan.

Characteristics:

Styles: From *Kond* to *Kond*.
1st level: K1 pattern, periodic. Tiles: P1, P2, $N1_1$, N2
2nd level: All [S1] tiles: P1, P2, $N1_1$, N2, N3.
Inflation rules: System 2, with an alternative option for the pattern of P1 (P1c in chapter 4, Figure17).
2-level pattern, globally periodic. Not Self-Similar because the inflation rule is not define for the tile N3 (*Sormedan*), and the first level, periodic, cannot be mapped onto the second one.
Particularities: The use of two mappings for the pentagons, P1c and its symmetric (1 and 2 on Figure 57-b). The orientations of those shapes (white arrows on Figure 57-b) is a little bit puzzling. In my opinion this cannot be an error from the artist, but a deliberate choice.

Can we define a Self-similar pattern from it, with some modifications? The problem is the use of two different mappings for the same pentagon. If we chose only one of them, the System 2 works easily.

a



b

Figure 57. Isfahan, Darb-e Imam.

Example 4, Figure 58: A tiling on a wall in Chahar Bagh Madrasa, Isfahan.
We have highlighted in white the basis of the figure. The mapping of the edges is different to what is used in systems 1 or 2: The stars on the edges are connected with a tile N6 in between. In black, the special tiles used on the mapping of P2 (The same as for the variations $N1_3$ and $N6_1$).

Characteristics:
Styles: From *Kond* to *Kond* + *Shol* (or from [S1] to [S]).
1st level: K1 pattern, again.
2nd level: All the tiles of [S] are present except P4 and P5. Variation $N1_1$, as usual, on both levels, variation $P6_1$, and an extra tile in the mapping of P2 (although there is a solution with no extra tiles).
Inflation rules: New system. The scale ratio is the same as for the system 2.
Two-level pattern, globally periodic. Not self-similar because the inflation rules are not defined for most of the tiles at the second level (P3, N3, N4, N5, N6 and the extra tiles). Moreover, as usual the periodic first level cannot be mapped onto the second.
Can we define a self-similar pattern from it, with some modifications?
I will leave to the reader the pleasure to work on this question.

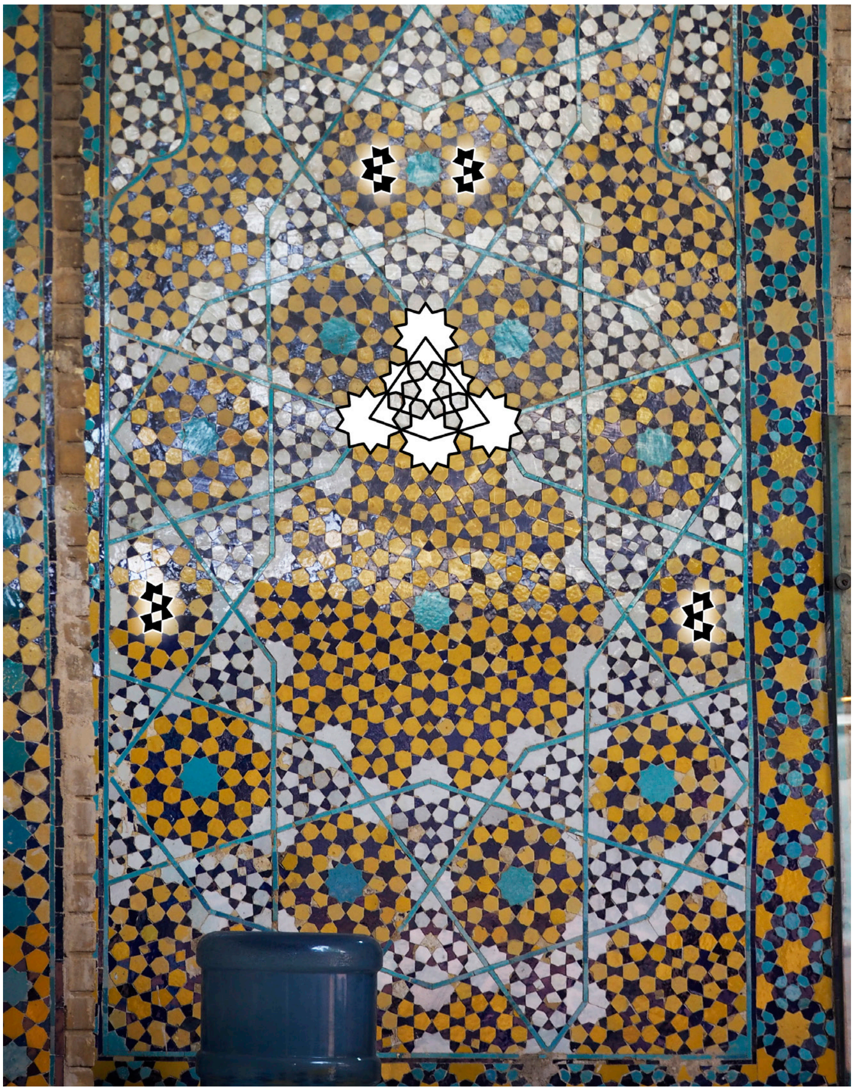


Figure 58. Isfahan, Chahar Bagh Madrasa. Sorry for the foreground, I could not avoid it.

At this point a question arise: Why artists haven't done a strict self-similar pattern? At least a pattern with a *Sormedan* at the first level, even though the whole pattern would be inserted into a periodic tiling?
I do not have an answer. Maybe there is such a pattern somewhere.

Example 5, Figure 59: Friday mosque, Isfahan. This is a rare example with a *Sormedan* at the first level (which contain only 2 tiles). The basis of the mapping is particular: The stars on the edges are oriented differently than usual, and are connected by a tile P2.
But the artists find difficulties in mapping the *Sormedan*. Their solution is quite tricky (in the skillful sense). This can be compared to Bonner's "Self-Similar" (in a weak sense) figure 13, on his web site.

Figure 59. Isfahan, Friday mosque

Characteristics:
Styles: From *Kond* to *Kond*.
1st level: Periodic pattern, tiles P1 and N3.
2nd level: Tiles P1, P2, $N1_1$, N2, N3 and exotics.
Inflation rules: New system, rule clearly defined only for the tile P1 (Pentagon).

7.6. Self-Similarity with interlaces.

The photography (Figure 60) shows a 2-level tiling with interlaces at the Madrasa Chahar Bagh, Isfahan.

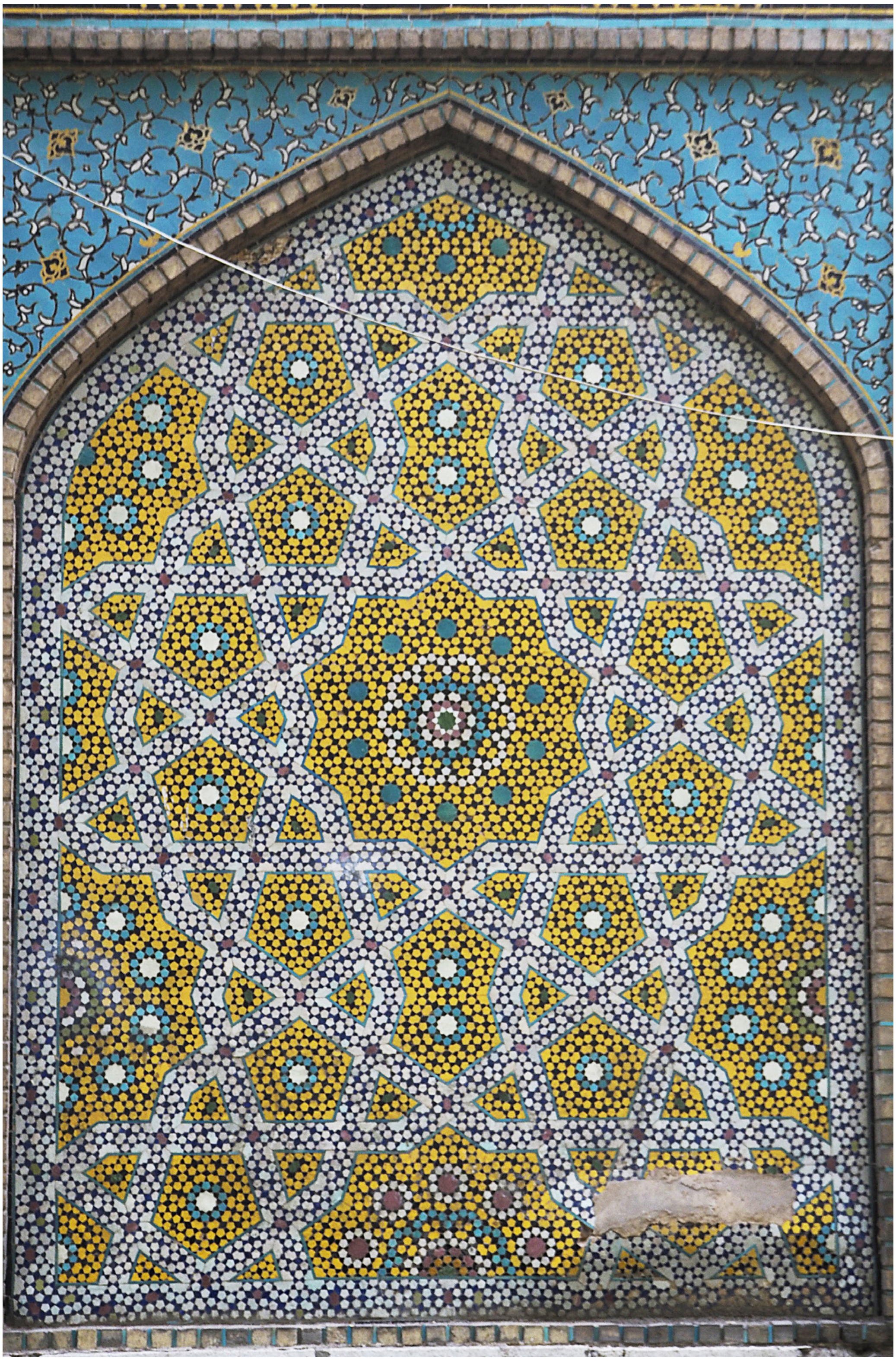
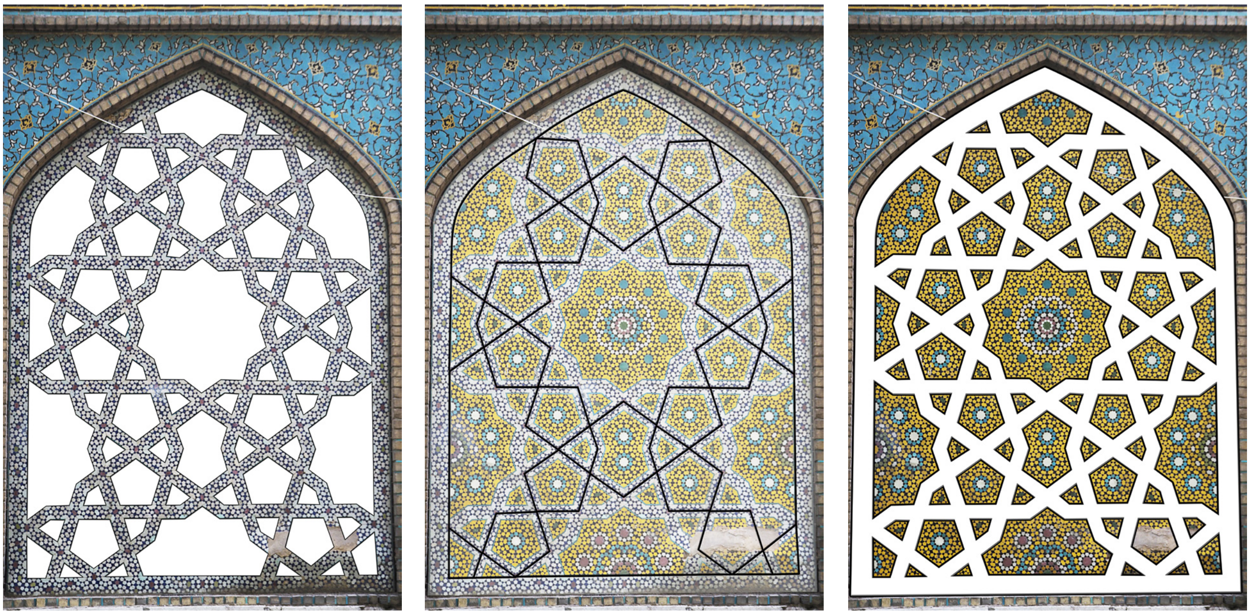


Figure 60. Isfahan, Chahar Bagh Madrasa.

The two levels belong to the *Kond* family (tiles from the set [S1]) but the tiling is not self similar in the strict sense because of the two usual reasons: the inflation rule is missing for the tile N3 (*Sormedan*), and the first level is periodic. In the next figure we have highlighted the basic pattern K1 (simplified), the interlaces, and p1, p2, n2... the inside shape of P1, P2, N2... after adding the interlaces.
Let's start the analyze by asking this question: can we find a self-similar *Kond* pattern with interlaces?



a b c

Figure 61. Middle, the basic "Mother of Tiling" pattern. Left the decorated interlaces, right the inside shapes.

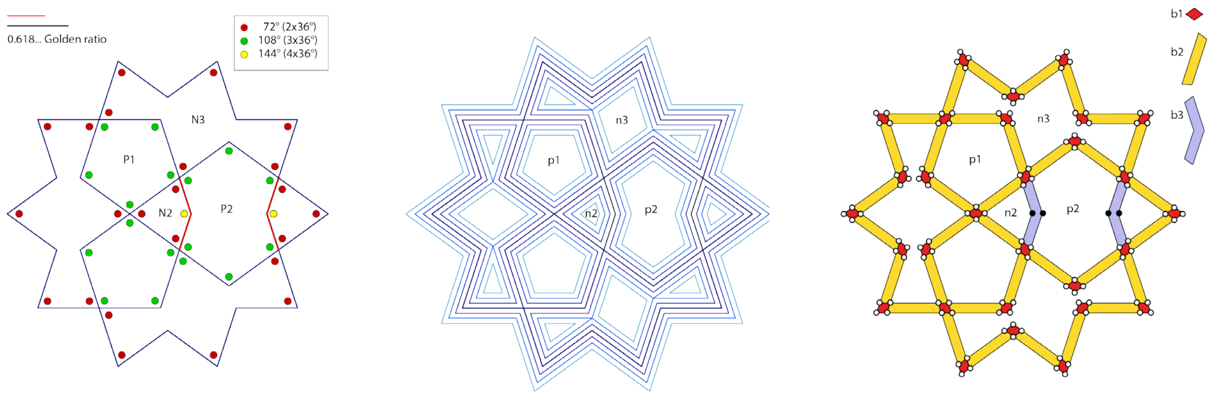


Figure 62. The tiles of a *Kond* pattern (a). Adding interlaces of different width (b). Components of a pattern with interlaces (c).

We can see on Figure 62-b that the shapes p2 and n3 have not the same proportions than the original P2 and N3, although the angles are the same. The proportions of p1 and n2 are still the same as P1 and N2, but they are no longer at the same scale. The length of the corresponding edges are different.
We start from the simple idea to have a star centered on each crossing point (white dots in Figure 62-c) and also on each point of discontinuity (black dots). Now, we have to fix the width of the interlaces, and fill the shapes p1, p2, n1, n2, n3 and the three additive shapes of the interlaces, b1, b2, b3.

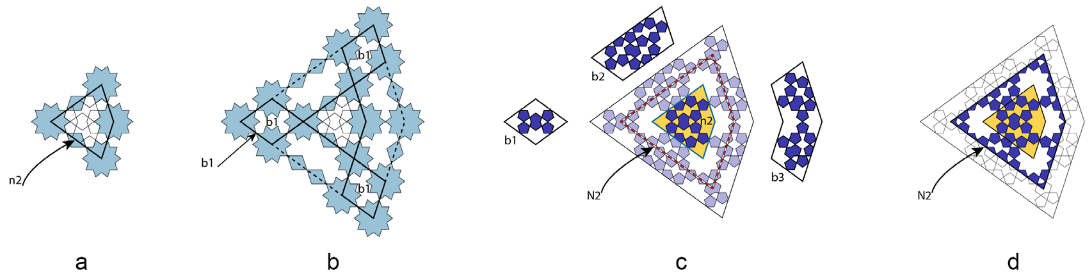


Figure 63. From n2 to N2, via the interlaces.

It makes sense to start from the smallest tile n2, every other will be inferred from it: In this story, small is beautiful, small is powerful!
The simplest solution consists in having connected stars along the small edge (Figure 63-a). This correspond to the mapping of N2 in our System 2 (chapter 4). Then we choose again the simplest solution for the tile b1 (Figure 63-b). Therefore we get the solution for each mapping of the three shapes of the interlaces (63-c). In fact, we can now get rid of all secondary shapes and consider only the mapping of N2, with the lines of the interlaces inside (Figure 63-d).
Then, we infer the mapping of the other tiles P1, P2, N1, N3 after drawing the half interlaces along their perimeter. Fortunately there is a solution, even for the tile N3 (Figure 64), for which the length of the interlaces is maximal.
Notice that, contrary to self-similar systems 1 and 2, there is nothing special for the *Sormedan* N3.

Figure 65 shows a detail of a 3-level pattern made from this system, which is a generalization of the Chahar Bagh tiling. We have used color inversion to highlight the interlaces of level 1 and 2.

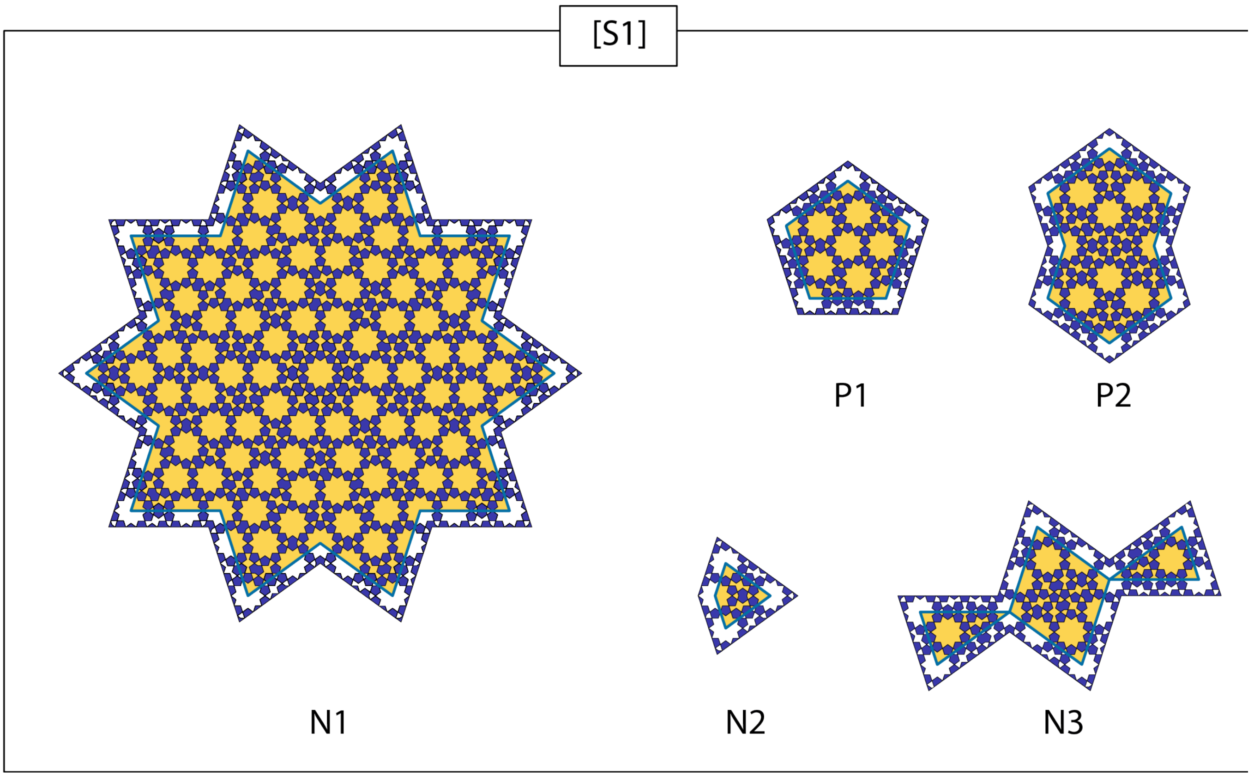


Figure 64. The set of mapped tiles of the self-similar system with interlaces. There is again two positive tiles, P1 and P2, and three negatives, N1, N2 and N3.

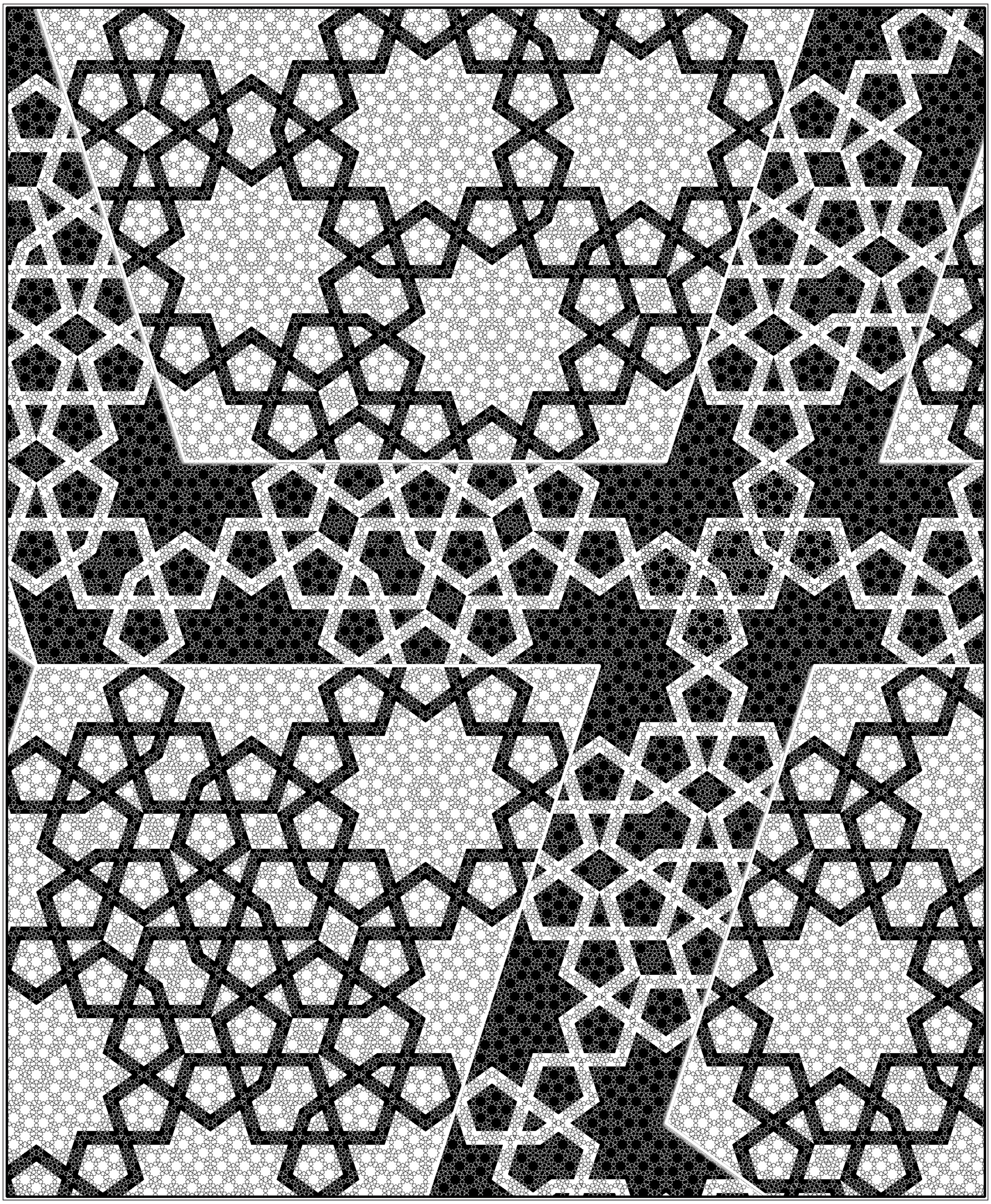


Figure 65. Three levels of a self similar pattern with interlaces (detail).

Conclusion

In my experience, it is often useful to experiment with alternative views on geometric patterns... or anything. Sometimes this can open new paths, provoke surprising connections, and stimulate creativity.
This work was starting from very simple questions. Then, on the way, one thing comes out and leads to another thing, and so on... Working on variations is a never-ending process!
So, more have to be done. About the traditional heritage, new points of view will certainly emerge in the future. Concerning pattern creation... Space is open.

In everything, my interest does not goes to the thing itself, but to its relationships with other things.

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I strongly recommend also the reading of every publication from Antony Lee, Peter Cromwell (available on http://www.liv.ac.uk/~spmr02/islamic/), the new book of Jay Bonner: Islamic Geometric Patterns: Their Historical Development and Traditional Methods of Derivation, once it's published, C. Kaplan's thesis, and to experiment the software "Taprats", from C. Kaplan as well.

Credits